CHOICE-BASED REVENUE MANAGEMENT WITH APPLICATIONS IN TRANSPORTATION

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CHOICE-BASED REVENUE MANAGEMENT WITH APPLICATIONS IN TRANSPORTATION

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It is our choices, Harry, that show what we truly are, far more than our abilities.

— J.K. Rowling, Harry Potter and the Chamber of Secrets

To my parents.

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SUMMARY

This dissertation consists of three studies on choice-based revenue management. All these studies are motivated by applications that arise in real-world business settings and have not been well addressed in the literature. The unifying theme of these studies is finding appropriate models to account for customer choice behavior while solving the corresponding revenue optimization problems effectively. Chapter 2 considers a network revenue management problem for airlines, where airline customers tend to purchase on price. Chapter 3 explores a dynamic load pricing problem in truckload marketplaces from market makers' perspective, where carriers exhibit choice behavior on loads. Chapter 4 studies assortment optimization problems in which some customers are rigid about the products they want while others are willing to substitute if their most preferred products are not available. We propose novel problem formulations, give efficient solution methods, derive insights in operational decisions, and develop near-optimal and executable strategies.

CHAPTER 1
INTRODUCTION

Over the past decades, discrete choice models have received significant attention in the research and practice of revenue management to capture customer choice behavior, giving rise to choice-based revenue management. As opposed to a traditional approach that assumes that each customer only purchases a predetermined product, choice-based revenue management accounts for the fact that customers substitute among available products or simply leave without purchasing if their preferred products are unavailable. It is not surprising that choice-based revenue management has been reported to provide significant improvements in revenue generation.

While various discrete choice models have been proposed and studied in the revenue management context, there is a gap between the fruitful findings in academia and industry practices. Companies who implement choice-based revenue management usually only adopt simple models, e.g., the multinomial logit choice model, or turn to black-box machine learning methods that are ready to apply but provide few managerial insights. A more sophisticated choice model may capture customer choice behavior more faithfully, but a more straightforward choice model typically results in tractable revenue optimization problems. This dilemma prohibits the advancement of choice-based revenue management in practice and many benefits it can provide.

This dissertation strives to narrow the gap by balancing between the sophistication of choice modeling and the tractability of revenue optimization. In particular, we formulate revenue management problems under discrete choice models motivated by evidence observed from real-world applications. We characterize structures of the optimal solutions, provide efficient solution methods or tractable approximations, derive key managerial insights, and develop near-optimal operational strategies.

Chapter 2 considers a revenue management problem under a spiked multinomial logit model. This study is motivated by our collaboration with an airline partner. We observe a prominent phenomenon in our airline partner's booking data that a disproportionate number of customers would purchase a product when it becomes the cheapest available one in the offered set of alternatives. The classic multinomial logit model cannot capture this phenomenon. Thus, we adopt a variant of the multinomial logit model, the spiked multinomial logit model, that accommodates the phenomenon, and formulate a network revenue management problem for airlines. Due to the "curse of dimensionality," we study a deterministic approximation of the problem. We show that the approximation can be solved efficiently by solving a small linear program. We use the solution of the small linear program to construct a booking limit policy and prove that the policy is *asymptotically optimal*: Roughly speaking, the policy achieves near-optimal performance when the market is large. To our knowledge, this is the first such result for a booking limit policy under a discrete choice model.

Chapter 3 considers a dynamic load pricing problem for truckload transportation marketplaces. The study is motivated by the emergence of online truckload marketplaces, where a market maker buys transportation services from carriers and sells transportation services to shippers. In these marketplaces, shippers often enter into long-term contracts with the market maker, and the market maker secures shipping capacity from carriers in a real-time market. Therefore, it is crucial for the market maker to adjust its offered price to carriers for each load based on the dynamics of supply (shipping capacity of carriers) and demand (load requests from shippers). While there are previous studies on truckload pricing problems, we notice few that consider carriers' random choice behavior, which is an essential part of the market dynamics. We use a multinomial logit model to incorporate carriers' random choice behavior and formulate the market maker's dynamic load pricing problem as a Markov decision process. We study a discrete-time fluid approximation of the problem and propose a simple pricing policy based on its solution. We show that the proposed policy is *asymptotically optimal* with a loss ratio of order $O(1/\theta)$, where θ represents the scale of shipper demand and carrier supply. This loss ratio is surprisingly lower than the usual $O(1/\sqrt{\theta})$ loss ratio reported in the revenue management literature. We also present a continuous-time fluid model and discuss the managerial insights provided by its solution.

Chapter 4 considers assortment optimization problems when customers choose under a mixture of multinomial logit and independent demand models. Using a mixture of these two models is arguably the most natural approach to combine the representational power of both models. It is also motivated by plausible purchase behavior: Some customers are rigid about the products they want, while others are willing to substitute if their preferred ones are not available. Assortment optimization problems under mixtures of choice models are notoriously difficult. So, it is surprising as we show that the single-shot assortment optimization problem under our mixture choice model is efficiently solvable. We provide a polynomial-sized linear program formulation and a combinatorial algorithm for solving the single-shot assortment optimization problem. To our knowledge, we are the first to give efficient methods for assortment optimization under a mixture of choice models. We also formulate an assortment-based network revenue management problem. We reduce a standard linear programming approximation of the problem with an exponential number of variables to an equivalent one of substantially smaller size.

CHAPTER 2
NETWORK REVENUE MANAGEMENT UNDER A SPIKED MULTINOMIAL LOGIT CHOICE MODEL

2.1 Overview

Revenue management is widely used by airlines to maximize revenues through optimizing inventory control and pricing. Most airlines have a number of fare classes for each itinerary, where an itinerary refers to a timed sequence of flights. Each fare class has certain booking rules (e.g., change fees and frequent flyer credits) and a price that for revenue management purposes is regarded as predetermined. Airlines then control prices by opening or closing fare classes. We refer to a combination of an itinerary and a fare class as a *product*. Thus, a basic decision in airline revenue management is to select which subset of products to offer to customers at each point in time. The subset of products made available to customers is called an assortment, and the problem of selecting a subset to maximize revenue is known as assortment optimization. Airlines dynamically adjust assortments based on the remaining seats on flights and the time until departure. The assortment decisions need to be considered jointly for the flights in an airline network, because some itineraries use capacity on multiple flights and because customers may substitute among different itineraries. The scale of airline networks makes solving assortment optimization problems challenging.

One of the most popular choice models is the multinomial logit (MNL) model. The MNL model has an easily interpretable structure, and the parameter estimation problem as well as the assortment optimization problem and the price optimization problem under the MNL model are tractable [1, 2, 3]. However, the MNL model falls victim to the independence from irrelevant alternatives (IIA) property, which states that the relative choice probabilities of two alternatives do not depend on the presence of other alternatives in the choice set. This can be severely restrictive for modeling the choice behavior of airline customers, among others for the following reason. It has been noticed that significantly more customers tend to choose the cheapest fare class among a considered set of available products than predicted by the MNL model, for example, most customers who book a ticket choose the cheapest available fare class for their chosen flight [4, 5]. This phenomenon violates the IIA property. For example, Figure 2.1 shows an airline's booking data for a specific flight. The fare classes are ordered such that Class 1 has the highest price and Class 8 has the lowest price. In the left figure, we show the fraction of bookings in each fare class for the flight when all eight fare classes are open. In the right figure, we show the fraction of bookings in each fare class for the flight when only Classes 1 to 7 are open. Note that in both cases, the cheapest available fare class (Class 8 on the left and Class 7 on the right) receives more than 60% of bookings. Moreover, the fraction of bookings in Class 7 is significantly more than that in Class 6 when Class 7 is the cheapest available fare class (on the right), but the fraction of bookings in Class 7 is less than that in Class 6 when Class 7 is not the cheapest available fare class (on the left). Since the ratio between the fractions of bookings in Class 7 and Class 6 is affected by the inclusion of other alternatives (such as Class 8), the IIA property is violated.

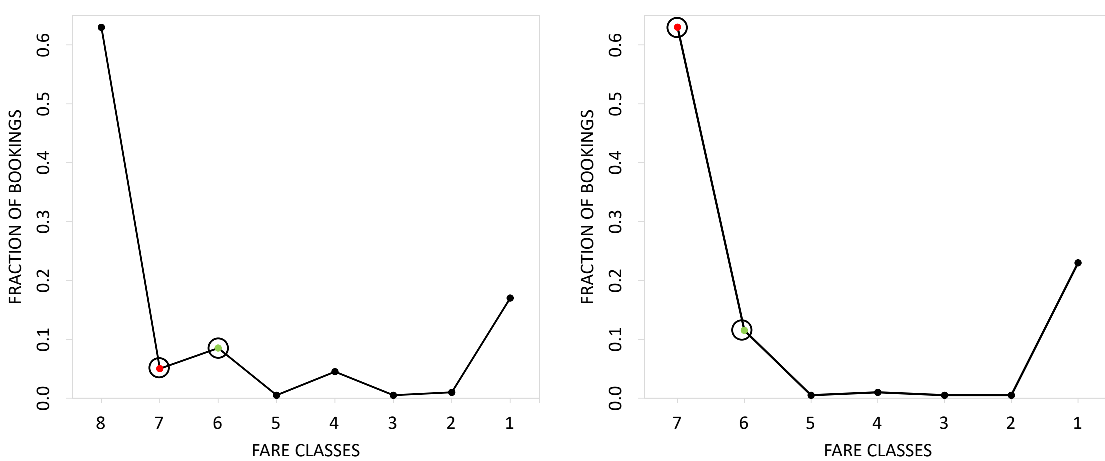

Figure 2.1: Historical booking data

We refer to the above phenomenon as "the spike effect," i.e., a disproportionate number of customers (than predicted by the MNL model) buy the cheapest available fare class on each flight. To capture the spike effect, we consider an extension of the MNL model that has separate preference weights for the cheapest products among the offered sets of available products. We call this the *spiked multinomial logit* (spiked-MNL) choice model, which was first introduced by [5].

Main Contributions: We characterize the structure of assortments that are Pareto-optimal, provide a tractable approximation, and show asymptotic optimality of a booking limit policy built on the solution of the approximation.

Assortment Optimization. Motivated by empirical data from airlines, we adopt a spiked-MNL model introduced previously and further study its properties. We formulate a dynamic assortment optimization problem under the spiked-MNL model as a Markov decision process. We establish that the efficient sets, which offer a Pareto-optimal trade-off between revenue and resource usage, are nested-by-revenue assortments under the spiked-MNL model (Theorem 2.4).

Linear Program Approximation. We consider a deterministic (fluid) approximation of the network revenue management problem under the spiked-MNL model, known as the choice-based deterministic linear program (CDLP). Even though this linear program has exponentially many variables, we prove that it can be solved in polynomial time by solving an equivalent sales-based linear program (SBLP) and by exploiting the nested-by-revenue structure of efficient sets (Theorem 2.7).

Asymptotic Optimal Policy. We show how the CDLP solution can be used to construct a nested booking limit policy of the form widely used in practice, and we prove the asymptotic optimality of such a booking limit policy (Theorem 2.8). To our knowledge, this is the first such result for booking limit policy in the assortment optimization setting. To deal with the randomness of assortments resulting from applying the booking limit policy, our proof uses an approach that is very different from those used for previous asymptotic optimality results.

2.2 Related Literature

There is an extensive body of literature on assortment optimization [6, 7]. Our literature review below focuses on papers that study assortment optimization for airlines under customer choice behavior, which are the most relevant for our work. The idea of airline assortment optimization can be traced back to traditional revenue management methodology such as Littlewood's proposal for controlling the availability of two fare classes [8]. Traditional demand models assume that each customer comes with a request for a predetermined product (flight and fare class combination). The seller then decides whether to accept or reject the customer's request. Revenue management under this traditional independent demand assumption is surveyed in [9]. As ignoring customer choice behavior may lead to cascading deterioration of revenue performance [10], some partial modeling remedies such as buy-downs and buy-ups, or spill-and-recapture, have been proposed to incorporate demand substitution [11, 12, 13].

More recently, choice-based demand models have been studied more widely [14]. The authors in [15] consider the problem of assortment optimization under a general choice model for a single flight, which is formulated as a dynamic program. Due to the curse of dimensionality, the computational burden of solving the dynamic program increases exponentially from a single flight to parallel flights, and to general airline networks. Therefore, a choice-based deterministic linear program (CDLP) is proposed in [16] as an approximation of the dynamic program. The solution of the CDLP can be used to build good control policies. For example, [17] extends the concept of efficient sets from [15] and proves that the policy based on the the CDLP solution is asymptotically optimal for the dynamic program. Even though the number of efficient sets usually is much less than the number of subsets, there could still be exponentially many efficient sets and thus exponentially many decision variables for the CDLP. Therefore, [17] also suggests solving the CDLP using column generation. Recently, [18] proposes a SBLP formulation for general attraction demand models, including the MNL model. The SBLP has a polynomial number of variables under the MNL model and is equivalent to the CDLP. For alternative approaches of approximating the dynamic program, we refer readers to [19] for approximation of the value functions, and [20] for a segment-based deterministic concave program.

In addition to assortment optimization under general choice models, many researchers have considered assortment optimization for more specific choice models. Among all these models, the MNL model is one of the most studied ones [1, 2, 17, 21, 22, 18]. The MNL model has many favorable properties, such as the maximum likelihood estimation problem, the assortment optimization problem, and the optimal pricing problem, being easy to solve. Also, some structure results on the optimal assortment or the pricing decision are presented under the MNL model. For example, the optimal policy of the assortment optimization problem under the MNL model is nested-by-fare-order for single flight revenue management [15]. Other choice models that have been considered in the assortment optimization literature include robust MNL [23, 24], nested logit [25, 26, 27], mixed [28, 29], Markov chain [30], and rank-based choice models [31, 32].

Typical airline reservation control systems use either booking limits [33, 34] or bid-prices [35] to control the availability of fare classes. Originally, these controls were motivated by the structure of optimal booking controls for single flights under the independent demand model. Although optimal booking controls can not be implemented in general with booking limits or bid prices, it nevertheless is of practical importance to find good booking controls that can be implemented with an airline's reservation control system.

The spike effect has been noticed in the airline industry before [4]. However, we are not aware of any research besides [5] and [36] that considers the spike effect in choice models. As shown in the example in Figure 2.1, the spike effect cannot be represented by the MNL choice model. Therefore, both [5] and [36] use an extension of the MNL model, i.e., the spiked-MNL model, to incorporate the spike effect.

2.3 Problem Formulation

We consider a revenue management problem with a network of flights, marketed by a single decision maker, which depart during a considered time period, such as one day. Let $\mathcal{F}$ denote the set of flights, and let $m := |\mathcal{F}|$ denote the number of flights. For each flight $f \in \mathcal{F}$, let c_f denote the seat capacity of flight f, and let $\boldsymbol{c} := (c_f, f \in \mathcal{F}) \in \mathbb{Z}_+^m$. An itinerary consists of a subset of flights, including a single flight or a sequence of connecting flights. Let $\mathcal{G}$ denote the set of itineraries. For each itinerary, the airline has a number of fare classes, each of which specifies a price and is associated with certain booking rules, e.g., refundability. A *product* is a combination of an itinerary and a fare class for that itinerary. Let $\mathcal{J}$ denote the set of products, and let $n := |\mathcal{J}|$ denote the number of products. For each flight $f \in \mathcal{F}$ and product $j \in \mathcal{J}$, let $a_f^j \in \{0, 1\}$ indicate whether product j uses a seat on flight f, and let $\boldsymbol{a}^j := (a_f^j, f \in \mathcal{F}) \in \{0, 1\}^m$. Let r_j denote the net revenue of product j, and let $\boldsymbol{r} := (r_j, j \in \mathcal{J}) \in \mathbb{R}^n$. For each itinerary $g \in \mathcal{G}$, let $\mathcal{J}^g$ denote the set of products for itinerary g, and let $n(g) := |\mathcal{J}^g|$, i.e., the number of fare classes for the itinerary.

The selling horizon is partitioned into discrete periods indexed by $t = 1, \ldots, T$. We assume that the time periods are sufficiently short so that there is at most one customer arrival in each period. In each period t, the airline selects an assortment $A(t) \subset \mathcal{J}$ to offer to customers. Each customer considers only a subset of the products in $\mathcal{J}$ for purchase. For example, a customer who wants to travel from origin A to destination B, considers only the products for itineraries that start at A and end at B. If a customer arrives in period t, let $C(t) \subset \mathcal{J}$ denote the consideration set of the customer. Let $j = 0$ denote the alternative always available to each customer that one can buy nothing from the airline, also called the no-purchase alternative or the null alternative. Let $S(t) := A(t) \cap C(t)$. Thus, if a customer arrives in period t, then the customer chooses an alternative in the customer's choice set $S(t) \cup \{0\}$. We assume that the collection of possible consideration sets form a partition of $\mathcal{J}$, such that each subset in the partition contains all the products for some subset of itineraries. For example, if each customer is interested in one origin-destination pair, then the consideration sets form a partition of $\mathcal{J}$ with each subset in the partition consisting of the products for itineraries for one origin-destination pair. Let $\mathcal{H}$ be an index set for the collection of consideration sets (for example, $\mathcal{H}$ denotes the set of origin-destination pairs in the network), and let $\{\mathcal{J}(h) : h \in \mathcal{H}\}$ denote the partition of $\mathcal{J}$. Then, for each t, $C(t) = \mathcal{J}(h)$ for some $h \in \mathcal{H}$. Each $h \in \mathcal{H}$ will also be called a market. Note that either all the products for an itinerary g serve a market h or none of the products for itinerary g serves market h, that is, if any product $j \in \mathcal{J}^g$ satisfies $j \in \mathcal{J}(h)$, then $\mathcal{J}^g \subset \mathcal{J}(h)$, and thus we sometimes write $g \in \mathcal{J}(h)$. In each period t, a customer arrives with consideration set $C(t) = \mathcal{J}(h)$ with probability $\lambda_h(t)$. With probability $1 - \sum_{h\in\mathcal{H}} \lambda_h(t)$, no customer arrives in period t. If a customer arrives in period t with consideration set $C(t) = \mathcal{J}(h)$, then the customer chooses product $j \in S(t)$ with probability $P^h_{j:S(t)}(t)$ or chooses the no-purchase alternative with probability $P^h_{0:S(t)}(t)$, where $P^h_{0:S(t)}(t) + \sum_{j\in S(t)} P^h_{j:S(t)}(t) = 1$. For assortment $A \subset \mathcal{J}$, alternative $j \in \mathcal{J} \cup \{0\}$, and flight $f \in \mathcal{F}$, let

$$P_{j:A}(t) \ := \ \sum_{h\in\mathcal{H}} \frac{\lambda_h(t)}{\sum_{h'\in\mathcal{H}} \lambda_{h'}(t)} P^h_{j:A\cap\mathcal{J}(h)}(t)$$

denote the conditional probability, given that a customer arrives in period t, that j is chosen if A is offered, let

$$Q_f(A,t) \ := \ \sum_{h\in\mathcal{H}} \frac{\lambda_h(t)}{\sum_{h'\in\mathcal{H}} \lambda_{h'}(t)} \sum_{j\in A} a^j_f P^h_{j:A\cap\mathcal{J}(h)}(t) \ = \ \sum_{j\in A} a^j_f P_{j:A}(t)$$

denote the expected seat capacity on flight f consumed per customer arrival if A is offered, and let

$$R(A,t) \ := \ \sum_{h\in\mathcal{H}} \frac{\lambda_h(t)}{\sum_{h'\in\mathcal{H}} \lambda_{h'}(t)} \sum_{j\in A} r_j P^h_{j:A\cap\mathcal{J}(h)}(t) \ = \ \sum_{j\in A} r_j P_{j:A}(t)$$

denote the expected revenue per customer arrival if A is offered. The arrival types (no arrival or type h arrival) in different time periods are independent. Also, given the assortments $A(t)$ in different time periods t, the customer choices in the time periods are independent.

Given the initial capacity $\boldsymbol{c}$ of each flight, the airline dynamically selects an assortment for each period t in order to maximize the expected total revenue. We present a dynamic programming model of the revenue management problem. Let $c_f(t)$ denote the remaining capacity of flight f at time t, and let $\boldsymbol{c}(t) := (c_f(t), f \in \mathcal{F}) \in \mathbb{Z}_+^m$. For any $\boldsymbol{c} \in \mathbb{Z}_+^m$, let $\mathcal{J}_{\boldsymbol{c}} := \{j \in \mathcal{J} \, : \, c_f \geq a_f^j \; \forall \; f \in \mathcal{F}\}$ denote the set of products that can be offered with remaining capacities given by $\boldsymbol{c}$. Let $V_t : \mathbb{Z}_+^m \to \mathbb{R}$ denote the optimal revenue-to-go function at time t. The optimality equation is given by

$$\begin{aligned} V_t(\boldsymbol{c}) &= \max_{A \subset \mathcal{J}_{\boldsymbol{c}}} \left\{ \sum_{h \in \mathcal{H}} \sum_{j \in A \cap \mathcal{J}(h)} \lambda_h(t) P^h_{j:A \cap \mathcal{J}(h)}(t) \left[r_j + V_{t+1}(\boldsymbol{c} - \boldsymbol{a}^j) \right] \right. \\ &\qquad \left. + \left[\sum_{h \in \mathcal{H}} \lambda_h(t) P^h_{0:A \cap \mathcal{J}(h)}(t) + 1 - \sum_{h \in \mathcal{H}} \lambda_h(t) \right] V_{t+1}(\boldsymbol{c}) \right\} \\ &= \max_{A \subset \mathcal{J}_{\boldsymbol{c}}} \left\{ \sum_{h \in \mathcal{H}} \sum_{j \in A \cap \mathcal{J}(h)} \lambda_h(t) P^h_{j:A \cap \mathcal{J}(h)}(t) \left[r_j - (V_{t+1}(\boldsymbol{c}) - V_{t+1}(\boldsymbol{c} - \boldsymbol{a}^j)) \right] \right\} \\ &\qquad + V_{t+1}(\boldsymbol{c}). \end{aligned} \tag{2.1}$$

The boundary conditions are $V_t(\boldsymbol{0}) = 0$ for all t and $V_{T+1}(\boldsymbol{c}) = 0$ for all $\boldsymbol{c} \in \mathbb{Z}_+^m$. In the remainder of the chapter, we omit the time index t, because time-dependence is easy to incorporate in the model and in the results at the cost of more cumbersome notation.

2.4 The Spiked-MNL Choice Model

In this section, we define the *spiked-MNL* choice model and discuss its properties. The spiked-MNL choice model is adopted from the modified MNL model in [5] and [36] to capture the effect of cheapest fare spikes. To simplify notation, in this section we omit the market index h, since each customer is associated with one market h.

For every product $j \in \mathcal{J}$, the model has two parameters $w_j > 0$, $v_j > 0$. The quantity w_j represents the *special preference weight* of product j when it is the cheapest available fare class for its itinerary; otherwise, product j has a *regular preference weight* of v_j. We assume that the cheapest fare spikes are nonnegative, i.e., $w_j \geq v_j > 0$ for all products $j \in \mathcal{J}$. This is consistent with the airline data that motivated this model. The preference weight of the null alternative is denoted with v_0 and is called the *null preference weight*.

Suppose the customer's choice set is $S \cup \{0\}$. Let $\mathbf{I}(j, S)$ denote the indicator such that $\mathbf{I}(j, S) = 1$ if j is the cheapest available fare class in S for its itinerary, and $\mathbf{I}(j, S) = 0$ otherwise. The spiked-MNL model specifies that product $j \in S$ is chosen with probability

$$P_{j:S} \;=\; \frac{v_j(1 - \mathbf{I}(j, S)) + w_j \mathbf{I}(j, S)}{v_0 + \sum_{j' \in S} \left[v_{j'}(1 - \mathbf{I}(j', S)) + w_{j'} \mathbf{I}(j', S)\right]}.$$

The probability that the customer chooses the null alternative is given by

$$P_{0:S} \;=\; \frac{v_0}{v_0 + \sum_{j' \in S} \left[v_{j'}(1 - \mathbf{I}(j', S)) + w_{j'} \mathbf{I}(j', S)\right]}.$$

Recall that in the MNL model, given a choice set $S \cup \{0\}$, a customer chooses product $j \in S$ or the null alternative with probability

$$P_{j:S} \;=\; \frac{v_j}{v_0 + \sum_{j' \in S} v_{j'}} \text{ and } P_{0:S} \;=\; \frac{v_0}{v_0 + \sum_{j' \in S} v_{j'}}$$

respectively. Therefore, in the spiked-MNL model, when $w_j = v_j$ for all $j \in \mathcal{J}$, the spiked-MNL model reduces to the MNL model.

We observed that the spiked-MNL model defined above fits airline booking data better than the MNL model. Figure 2.2 provides a typical example on using the MNL and the spiked-MNL models to predict customer bookings . Specifically, in this example presented by Figure 2.2, seven fare classes are open on a given flight. The solid black line corresponds to the the actual fractions of bookings of the seven fare classes (conditioned on a booking of this flight), as well as the predicted fractions of the MNL and the spiked-MNL models calibrated with the same data. Clearly, the prediction of the spiked-MNL model is much closer to the actual data compared to the prediction of the MNL model. We refer readers to [5] for more examples and a discussion on the prediction performance of the spiked-MNL model compared with other models, including the MNL model.

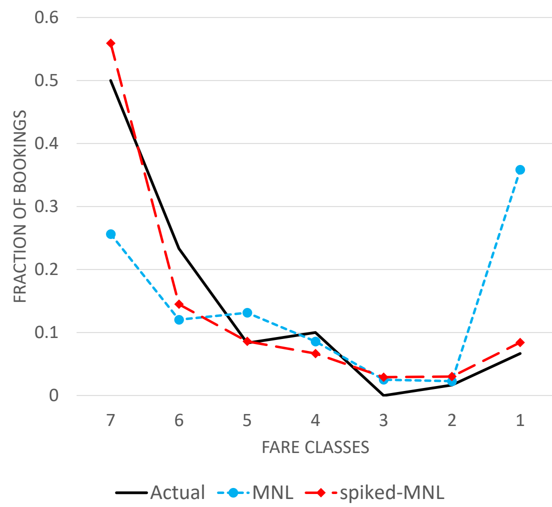

Figure 2.2: Actual and estimated fractions of bookings under the MNL and the spiked-MNL models

We would like to emphasize the difference between the MNL model and the spiked-MNL model in the context of random utility model. Under the MNL model, the utility that a customer associates with product j is given by $U_j = V_j + \epsilon_j$, where V_j is the deterministic component of the utility, and ϵ_j is an i.i.d. Gumbel random variable that introduces heterogeneity in the customer population. The deterministic component of the utility V_j is often written as a linear combination of the individual feature values of product j. In contrast, the spiked-MNL model assumes that the deterministic component V_j depends not only on the features of the product itself, but also on the features of the other products that potentially could be offered alongside product j. In our definition of the spiked-MNL model, the deterministic component of the utility is affected by the prices of other products; namely, if product j is the cheapest among a group of products offered, i.e., fare classes on the the same itinerary, its deterministic utility component V_j receives an additional spike. We also provide discussions on several properties of the spiked-MNL model in Appendix A.1.

2.5 Efficient Assortments under the Spiked-MNL Model

In some settings, an optimal policy offers only assortments, called *efficient sets*, that are Pareto optimal in terms of expected revenue and expected resource consumption. For example, for single flight revenue management, an optimal policy offers efficient sets only, and furthermore that under the MNL choice model, the efficient sets are nested-by-revenue [15]. If each product uses a unit of capacity on one flight, as in the case of parallel flight revenue management, then an optimal policy uses efficient sets only [17]. However, an optimal policy does not in general use efficient sets only, because the displacement cost $V_t(\boldsymbol{c}) - V_t(\boldsymbol{c} - \boldsymbol{a}^j)$ in (2.1) is not in general linear in $\boldsymbol{a}^j$.

In this section, we show that under the spiked-MNL model, efficient sets are nested-by-revenue. That is, if a fare class for an itinerary is open, then all fare classes for the same itinerary with higher fares are also open. In Section 2.6 we consider a fluid approximation of the network revenue management problem. For such problems, optimal solutions use efficient sets only. Therefore, optimal solutions of the fluid approximation under the spiked-MNL model offers only assortments that are nested-by-revenue.

2.5.1 Efficient Sets

First, we review the concept of efficient sets. In this section the time index t is omitted, since the concept will be applied to each t.

Definition 2.1 (Efficient Sets)**.** *An assortment $S \subset \mathcal{J}$ is said to be* inefficient *if a mixture of other assortments has strictly higher expected revenue with the same or lower expected resource consumption. That is, there exists a set of weights $\{\mu(A) : A \subset \mathcal{J}\}$ satisfying*

$\sum_{A \subset \mathcal{J}} \mu(A) = 1$ *and* $\mu(A) \geq 0$ *for all* $A \subset \mathcal{J}$ *such that*

$$R(S) < \sum_{A \subset \mathcal{J}} \mu(A) R(A) \qquad \textit{and} \quad Q_f(S) \geq \sum_{A \subset \mathcal{J}} \mu(A) Q_f(A) \quad \textit{for all } f \in \mathcal{F}.$$

An assortment that is not inefficient is said to be efficient.

The following lemma gives a necessary condition for an assortment to be efficient, and will be used to prove Theorem 2.4.

Lemma 2.2. *If an assortment* S *is efficient, then there exists a* $\boldsymbol{\gamma} \in \mathbb{R}^n$*, satisfying* $\gamma_j > \gamma_{j'}$ *for all* $j \in \mathcal{J}$ *and* $j' \in \underline{J}(j)$*, such that* S *is an optimal solution of the problem*

$$\max_{A \subset \mathcal{J}} \left\{ \sum_{j \in A} \gamma_j P_{j:A} \right\}. \tag{2.2}$$

2.5.2 Nested-by-Revenue Assortments

For single flight revenue management under the MNL model, the efficient sets are nested-by-revenue, that is, if an efficient set contains a product j, then it also contains all products with higher fares than j [15]. Moreover, the efficient sets are nested-by-revenue under the MNL model even when the model parameters are uncertain [24]. Below we give the natural extension of the concept of assortments that are nested-by-revenue for network revenue management, and we show that the efficient sets under the spiked-MNL model are nested-by-revenue.

Definition 2.3 (Nested-by-Revenue Assortments)**.** *An assortment* S *is* nested-by-revenue *if for any product* $j \in S$*, all products associated with the same itinerary as* j *and with higher revenues than* j *are also in the assortment. That is,* S *is nested-by-revenue if for any* $j \in S$*, it holds that* $J(j) \subset S$*.*

Theorem 2.4. *Every efficient set under the spiked-MNL model is nested-by-revenue.*

The theorem implies that if we restrict our attention to efficient sets, the number of possible assortments can be significantly reduced: for an itinerary $g \in \mathcal{G}$ that has $n(g)$ fare classes, rather than considering all $2^{n(g)}$ possible combinations, we only need to consider the $n(g)$ nested-by-revenue fare class combinations. However, we note that the total number of efficient sets is still exponential in the number of itineraries. In Section 2.6, we will show that the structural result from Theorem 2.4 leads to a tractable linear programming formulation of the fluid approximation of the dynamic program, whose size is polynomial in the number of itineraries.

Remark 2.5. *Recall that we assume the spike effect is nonnegative throughout the chapter, that is, $w_j \geq v_j$. If $w_j < v_j$, then the result of Theorem 2.4 does not hold in general. See the following counterexample.*

Example 2.6. *A seller sells three products: H, M, and L, with revenues $r_H = 5$, $r_M = 3$, and $r_L = 2$, using the same resource. The preference weights are $v_H = 5$, $v_M = 10$, $w_H = 2$, $w_M = 4$, and $w_L = 1$ (we don't need to specify v_L); the null preference weight is $v_0 = 10$. Figure 2.3 shows the plot of $(Q(S), R(S))$ and its convex envelope. This convex envelope represents the Pareto-optimal frontier. By Definition 2.1, all efficient sets are on the Pareto-optimal frontier. Note that assortment $\{H, L\}$ is on the Pareto-optimal frontier and hence an efficient set, but it is not nested-by-revenue.*

2.6 Deterministic Approximation and Static Booking Limit Control

The dynamic program (2.1) is intractable for large networks due to the curse of dimensionality. This motivates us to consider a fluid approximation of the dynamic program. A fluid approximation often used in the revenue management literature is the choice-based deterministic linear program (CDLP), that we present in Section 2.6.1. For both general choice models and the spiked-MNL model, the CDLP has exponentially many variables. In Section 2.6.2 we present a compact formulation which is equivalent to the CDLP and has only

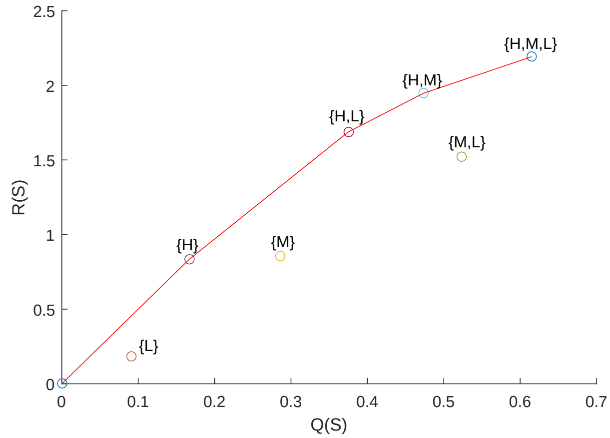

Figure 2.3: Efficient sets under a spiked-MNL model with $w_j < v_j$

n variables. Solutions of the compact formulation can be used to construct various booking control policies, including static booking limit policies that are studied in Section 2.7.

2.6.1 Choice-Based Deterministic Linear Program

The choice-based deterministic linear program (CDLP) is an approximation of dynamic program (2.1) in which customer arrivals and choices are replaced by their means, and capacity and demand are modeled as real-valued rather than integer valued [16]. Let decision variable $\alpha(A)$ denote the fraction of time that assortment $A \subset \mathcal{J}$ is offered, and let $\boldsymbol{\alpha} := (\alpha(A), A \subset \mathcal{J})$. The CDLP is given by

$$z^{\text{CDLP}} := \max_{\boldsymbol{\alpha} \geq 0} \sum_{A \subset \mathcal{J}} \alpha(A) T \sum_{h \in \mathcal{H}} \lambda_h \sum_{j \in A} r_j P^h_{j:A \cap \mathcal{J}(h)} \tag{2.3a}$$

$$\text{s.t.} \sum_{A \subset \mathcal{J}} \alpha(A) \leq 1, \tag{2.3b}$$

$$\sum_{A \subset \mathcal{J}} \alpha(A) T \sum_{h \in \mathcal{H}} \lambda_h \sum_{j \in A} a^j_f P^h_{j:A \cap \mathcal{J}(h)} \leq c_f \qquad \forall f \in \mathcal{F}. \tag{2.3c}$$

The objective (2.3a) of the CDLP is the expected total revenue over the horizon. Constraint (2.3b) specifies that the sum of the fractions of time that different assortments are offered is less than 1. In the remaining $1 - \sum_{A \subset \mathcal{J}} \alpha(A)$ fraction of time, all the fare classes are closed and the company offers an empty set. Constraint (2.3c) enforce seat capacity constraints.

The authors of [17] showed that optimal solutions of problem (2.3) use efficient sets only. Theorem 2.4 established that every efficient set under the spiked-MNL model is nested-by-revenue. Thus, under the spiked-MNL model, if assortment A is not nested-by-revenue, then decision variable $\alpha(A)$ can be omitted. Therefore the number of decision variables is reduced from $2^n = 2^{\sum_{g \in \mathcal{G}} n(g)} = \prod_{g \in \mathcal{G}} 2^{n(g)}$ to $\prod_{g \in \mathcal{G}} \big(n(g)+1\big)$, where $n(g)$ is the number of fare classes on itinerary $g \in \mathcal{G}$. However, the reduced number of decision variables still is exponential in the number of itineraries. This motivates us to develop an equivalent LP formulation with a polynomial size in the next section.

2.6.2 Sales-Based Linear Program

Under the MNL model, there is an LP formulation called the sales-based linear program (SBLP), which has a polynomial number of variables and constraints, and which is equivalent to the CDLP [18]. An SBLP formulation is developed for the spiked-MNL model in [5]. Next we use the result of Theorem 2.4 to develop an SBLP formulation for the spiked-MNL choice model that has fewer variables and constraints than its counterpart in [5].

Our SBLP formulation takes into account that all assortments offered by an optimal solution are nested-by-revenue. Let x_j denote the sales of product j when j is the cheapest available product for its associated itinerary, and let $\boldsymbol{x} := (x_j, j \in \mathcal{J})$. Consider any assortment A that is nested-by-revenue, and such that j is the cheapest available product in A for its associated itinerary. Note that $\bar{J}(j)$, the set of products associated with the same itinerary as product j and that have equal or higher fares than product j, satisfies $\bar{J}(j) \subset A$. Then, for the market h such that $j \in \mathcal{J}(h)$ and for any $j' \in J(j)$, it holds that $P^h_{j':A\cap\mathcal{J}(h)}/P^h_{j:A\cap\mathcal{J}(h)} = v_{j'}/w_j$, which is the same for all A satisfying the conditions above. Therefore, at the same time that x_j units of product j is sold when j is the cheapest available product for its associated itinerary, $(v_{j'}/w_j)\, x_j$ units of each product $j' \in J(j)$ is sold. Let x_0^h denote the number of customers in market h who choose the no-purchase alternative, and let $\boldsymbol{x}_0 := (x_0^h, h \in \mathcal{H})$.

The SBLP under the spiked-MNL model is given by

$$
\begin{aligned}
z^{\mathrm{SBLP}} \;=\; \max_{\boldsymbol{x},\boldsymbol{x}_0} & \sum_{h\in\mathcal{H}} \sum_{j\in\mathcal{J}(h)} \left(r_j + \sum_{j'\in J(j)} r_{j'} \frac{v_{j'}}{w_j} \right) x_j && && \text{(2.4a)}\\
\text{s.t. } & x_0^h + \sum_{j\in\mathcal{J}(h)} \left(1 + \sum_{j'\in J(j)} \frac{v_{j'}}{w_j} \right) x_j \;=\; \lambda_h T && \forall\, h \in \mathcal{H} && \text{(2.4b)}\\
& \sum_{h\in\mathcal{H}} \sum_{j\in\mathcal{J}(h)} \left(1 + \sum_{j'\in J(j)} \frac{v_{j'}}{w_j} \right) a_f^j x_j \;\le\; c_f && \forall\, f \in \mathcal{F} && \text{(2.4c)}\\
& \sum_{j\in\mathcal{J}^g} \frac{x_j}{w_j} \;\le\; \frac{x_0^h}{v_0} && \forall\, h \in \mathcal{H}, g \in \mathcal{J}(h) && \text{(2.4d)}\\
& \boldsymbol{x} \;\ge\; \mathbf{0},\; \boldsymbol{x}_0 \;\ge\; \mathbf{0}.
\end{aligned}
$$

The objective (2.4a) is the total revenue. Constraint (2.4b) represents the fact that, for each market, the number of bookings plus the number of no-purchase customers equals the number of arrivals. Constraint (2.4c) is the seat capacity constraint for each flight. Constraint (2.4d) is a generalization of the scale constraint in [18] to include the spike effect. The quantity x_j/w_j is proportional to the amount of time that product j is the cheapest available product for its associated itinerary. Since the null alternative is always available, the constraint states that the total amount of time that different products are the cheapest available products for an itinerary cannot exceed the total amount of time that the null alternative is available. The SBLP formulation above applies to a time-homogeneous demand model and a single booking channel. It is easy to extend the SBLP formulation to a piecewise constant time-varying demand model and multiple booking channels. This extension is used in our numerical experiments based on real-world airline data (Section 2.8).

The following result establishes that the SBLP formulation (2.4) is equivalent to the

CDLP formulation (2.3) under the spiked-MNL model.

Theorem 2.7. *Under the spiked-MNL model, given an optimal solution of the CDLP* (2.3)*, an optimal solution of the SBLP* (2.4) *can be constructed in polynomial time, and vice versa. Moreover, the CDLP* (2.3) *has an optimal solution that consists of a nested sequence of assortments, each of which is nested-by-revenue.*

The proof of Theorem 2.7 is constructive: we give an algorithm that converts optimal solutions between the two formulations in polynomial time. In addition, the algorithm is designed to produce an optimal CDLP solution that consists of a *nested sequence of assortments*, each of which is nested-by-revenue. That is, the algorithm constructs a sequence of assortments $S_1 \supset S_2 \supset \cdots \supset S_k$, each of which is nested-by-revenue, and an optimal CDLP solution $(\alpha(S), S \subset \mathcal{J})$, such that $\alpha(S) > 0$ only if $S \in \{S_1, \ldots, S_k\}$.

Note that, unlike the case for a single flight, for a network of flights, a set of assortments that are nested-by-revenue might *not* form a nested sequence — a simple counterexample is the following two nested-by-revenue assortments for two parallel flights: one assortment contains the highest fare class for the first flight only, and the other contains the highest fare class for the second flight only. The observation that there exists an optimal CDLP solution that consists of a nested sequence of assortments plays an important role in the construction of static booking limit controls that we discuss next.

2.7 Static Booking Limit Controls

Booking limits are widely used by airline reservation systems for controlling availability of fare classes. With a *partitioned* booking limit policy, a number of seats, called the partitioned booking limit (or just booking limit), is allocated to each product, and a product is closed for bookings once the number of units of that product sold reaches its booking limit. With a *nested* booking limit policy, a number of seats, called the nested booking limit, is allocated to each subset $\underline{J}(j) \cup \{j\}$, $j \in \mathcal{J}$, of products that is nested-by-revenue.

A nested booking limit policy can be implemented using either standard nesting or theft nesting [15, 34]. Under standard nesting, product j is closed if, for any product $j' \in \bar{J}(j)$, it holds that the number of units sold in subset $\underline{J}(j') \cup \{j'\}$ has reached the nested booking limit of that subset. Under theft nesting, product j is closed if the total number of units of all products for the associated itinerary sold has reached the booking limit of subset $\underline{J}(j) \cup \{j\}$. Under both nested booking limit policies, a higher revenue product is available whenever a lower revenue product is available. If there were no cancellations, then under any of the three booking limit policies above, once a product is closed, it would remain closed until the end of the horizon.

2.7.1 Booking Limits from the SBLP Solution

By Theorem 2.7, an optimal solution for SBLP (2.4) can be used naturally to obtain booking limits, where the booking limit for each product or each nested subset of products is given by the optimal sales of that product or that nested subset of products for the SBLP (2.4). In particular, let $\boldsymbol{x}^* = (x_j^* : j \in \mathcal{J})$ be an optimal solution for SBLP. The resulting amount of product j sold, denoted by b_j^*, is given by

$$b_j^* \;=\; x_j^* + \sum_{j' \in \underline{J}(j)} \frac{v_j}{w_{j'}} x_{j'}^*. \tag{2.5}$$

By (2.4c), $\sum_{j \in \mathcal{J}} a_f^j b_j^* \leq c_f$. We thus define a (static) partitioned booking limit policy by setting the booking limit of product j to b_j^*. We also define a (static) nested booking limit policy, where the booking limit b_j^{nested} for subset $\underline{J}(j) \cup \{j\}$ is given by

$$b_j^{\text{nested}} \;=\; \sum_{j' \in \underline{J}(j) \cup \{j\}} b_{j'}^*. \tag{2.6}$$

In this way, an optimal solution for SBLP (2.4) provides three static booking limit policies:

- a partitioned booking limit policy, using the sales given by (2.5) as booking limits;

- a standard nested booking limit policy, using booking limits given by (2.6);
- a theft nested booking limit policy, also using booking limits given by (2.6).

Under any of the static booking limit policies, and under any sample path of customer arrivals and choices, a sequence of assortments $S_1, S_2, \ldots, S_K$ are offered such that $S_1 \supset S_2 \supset \cdots \supset S_K$. If all the random variables in the system associated with customer arrivals and choices were replaced by their expectations, then the resulting sequence of assortments would correspond to an optimal CDLP solution, arranged to form a nested sequence of assortments (Theorem 2.7).

2.7.2 Asymptotic Optimality of the Static Partitioned Booking Limit Policy

We study the asymptotic properties of the partitioned booking limit policy defined above. In the asymptotic setting, it is convenient to consider the continuous time version of the problem. Thus, in this section, we assume that customers arrive according to a Poisson process instead of a Bernoulli process; the Bernoulli process considered in Section 2.3 can be viewed as an approximation of the Poisson process if the probability that more than one customer arrives in a period is negligible. We study the partitioned booking limit policy under the following asymptotic regime often considered in the revenue management literature. Let $\boldsymbol{c}$ denote the baseline capacity, and let λ denote the baseline arrival rate. Consider a sequence of revenue management problems indexed by $\theta = 1, 2, \ldots$, with capacity $\theta\boldsymbol{c}$ and arrival rate $\theta\lambda$, respectively. Other model parameters remain constant when θ grows. We refer to an revenue management problem scaled by θ as the *θ-scaled problem*. Let z_{OPT}^{θ} denote the optimal expected revenue for the θ-scaled problem. Note that for the θ-scaled problem, the optimal objective value of the corresponding CDLP is θz^{CDLP}, where z^{CDLP} denotes the optimal objective value of the baseline CDLP (2.3). Then $z_{OPT}^{\theta} \leq \theta z^{\mathrm{CDLP}}$. Let Z^{θ} denote the objective value, that is the (random) revenue, for the θ-scaled problem under the partitioned booking limit policy. The following result establishes that the partitioned booking limit policy is asymptotically optimal under fluid scaling.

Theorem 2.8. *The expected revenue* $\mathsf{E}[Z^\theta]$ *of the partitioned booking limit policy defined by* (2.5) *satisfies*

$$\lim_{\theta\to\infty} \frac{\mathsf{E}[Z^\theta]}{\theta} \;=\; z^{\text{CDLP}}.$$

Theorem 2.8 implies that

$$\lim_{\theta\to\infty} \frac{\mathsf{E}[Z^\theta]}{z^\theta_{OPT}} \;=\; \lim_{\theta\to\infty} \frac{\mathsf{E}[Z^\theta]}{\theta z^{\text{CDLP}}} \;=\; 1.$$

Therefore, when customer demand and seat capacities are large, the partitioned booking limit policy is near optimal.

The proof of Theorem 2.8 can be found in Appendix A.5. Importantly, our proof does not use the standard asymptotic optimality for the time-based CDLP-based policy [17] due to the following reason. The time-based policy divides the time horizon into intervals according to the CDLP solution, and offers a *fixed* assortment in each interval. The analysis is relatively simple, because the customer choices are i.i.d. in each interval. In contrast, the customer choices under a booking limit policy are more complicated: earlier customer choices can cause products to close, which affect later assortments, which in turn affect later customer choices. This complication does not exist for the independent demand model, for example, in the analysis of a static booking limit policy under the independent demand model in [37]. Therefore, a new technique was used to prove Theorem 2.8.

Our proof is based on the following approach. First, using the CDLP solution, we divide the time horizon into intervals given by time points $t_0 < t_1 < \cdots < t_k$. Then, we add some "padding" around each time t_i, given by $t_i^- = t_i - \nu_i^-\varepsilon$ and $t_i^+ = t_i + \nu_i^+\varepsilon$ for some small $\varepsilon > 0$ (see Figure 2.4). We show that with high probability, the assortments offered by the booking limit policy outside the intervals (t_i^-, t_i^+) are the same as the assortments offered by the CDLP solution. The booking process within intervals (t_i^-, t_i^+) can be complex, but we derive upper and lower bounds on the deviation of the booking process from the CDLP prediction. We show by induction that the booking quantities at the end of the time horizon is $O(\varepsilon)$ away from the static booking limit given by (2.5), and since we can choose ε to be arbitrarily small as $\theta \to \infty$, this proves the asymptotic result.

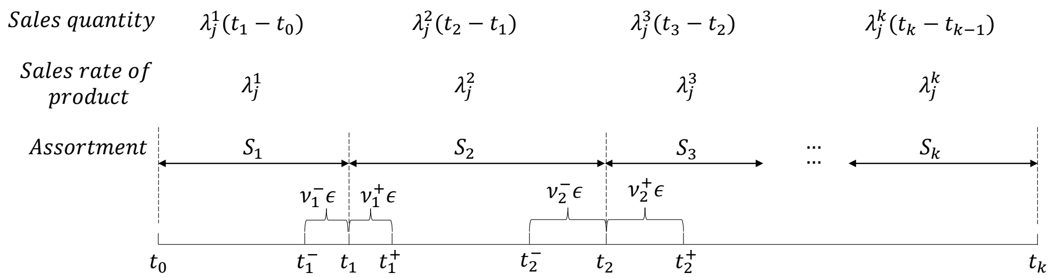

Figure 2.4: Segments of the time horizon for asymptotic analysis

2.8 Numerical Experiments

In this section, we first demonstrate the hazard of using an MNL model when the true underlying model is actually spiked-MNL. Then we compare the revenue performance under different booking control policies using real-world airline data.

2.8.1 Comparison between the MNL and the Spiked-MNL Models

We first consider a simple one-resource two-product example, where we demonstrate the evolution of revenue performance when we gradually learn consumer choice behavior and set controls accordingly. We observe a non-negligible gap in revenue between using an MNL model and a spiked-MNL model when the true underlying choice model is spiked-MNL.

Evolution of average revenue. A seller can offer two products H and L, which use the same resource and have price $r_H = 3$ and $r_L = 2$. We assume that over each selling season (one iteration in the simulation), the expected number of arriving customers is $\lambda T = 1000$. Customers make their choices according to a true underlying choice model with probabilities $P_{H:\{H,L\}} = 1/4$, $P_{L:\{H,L\}} = 1/2$ and $P_{H:\{H\}} = 2/3$. Here we only consider nested-by-revenue assortments as they are the efficient sets under the MNL and the spiked-MNL models. We control the fractions of time offering assortments $\{H, L\}$ and $\{H\}$. More specifically, we offer assortment $\{H, L\}$ for a fraction α_{HL} of the selling season, and then we offer assortment $\{H\}$ for the remaining time until we run out of stock or the selling season ends. Initially, we set the fraction of time offering $\{H, L\}$ as $\alpha_{HL} = 0.5$ and run iterations of simulation to get sales data. Then we estimate both MNL and spiked-MNL choice models using maximum likelihood estimation with the data generated so far and solve the CDLP to get the optimal control on α^*_{HL}. We update our control using $\alpha^{new}_{HL} = (1-\kappa)\alpha^{old}_{HL} + \kappa\alpha^*_{HL}$. In the equation, parameter $\kappa \in (0, 1)$ is a learning rate on control and we set $\kappa = 0.05$; α^{old}_{HL} is the control we use in the current iteration of simulation; α^{new}_{HL} is the control we will use in the next iteration. We run the simulation for 100 iterations and keep track of the revenue collected in each iteration. This gives us a trajectory of evolving revenue over one simulation run. We conduct multiple simulation runs and calculate the trajectory of average revenue.

Figure 2.5 shows the trajectories of average revenue over 200 simulation runs under the MNL and the spiked-MNL model, with different settings of capacity. The plot on the left shows a case where the average revenue under the MNL model deteriorates over time while the average revenue under the spiked-MNL model increases. The plot on the right shows a case where the average revenue under the MNL model slightly increases over time. However, there is a non-negligible gap between the average revenue under the MNL model and that under the spiked-MNL model.

Revenue performance with an airline dataset. We calibrate both the MNL model and the spiked-MNL model with a real-world airline dataset and examine the revenue performance under both models. The dataset is provided by an airline, containing an important origin-destination market with more than 30 parallel flights per day. In this market, the host airline operates 20 of these flights on each day, and each flight has the same fare class structure with 13 fare classes. There are 5 booking channels. The selling horizon is divided into 200 intervals. We consider different customer segments by allowing the choice parameters

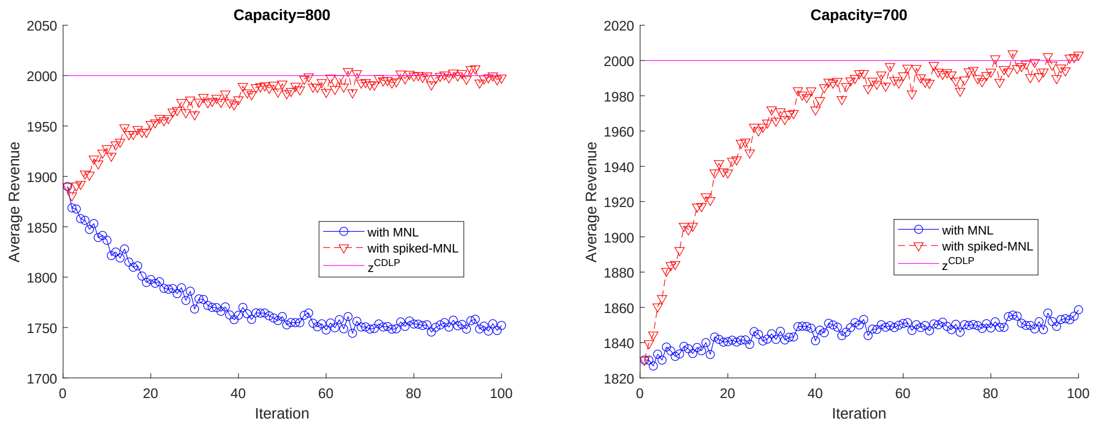

Figure 2.5: Trajectories of average revenues under the MNL and the spiked-MNL models

to vary along the booking horizon and across different booking channels. That is, within each interval-channel combination, we have a separate customer segment.

We model and estimate customer demand as follows. Let $\mathcal{N}$ denote the set of booking requests, including those who requested bookings with the host airline and other airlines, for any of the parallel flights on a specific departure date. Request $\tau \in \mathcal{N}$ arrives via channel c_τ in interval ℓ_τ, and is offered an assortment S_τ by the host airline. Let $\boldsymbol{x}^{\tau,j}$ denote a feature vector containing information about request τ and product j. For example, product-specific features include price, change fees, and frequent flyer mileage gain; request-specific features include booking channel and booking time. In the spiked-MNL model, $\boldsymbol{x}^{\tau,j}$ contains a binary variable indicating whether product j is the cheapest available fare class on the associated flight. Let $v(\boldsymbol{x}^{\tau,j})$ denote the preference weight of product j given feature vector $\boldsymbol{x}^{\tau,j}$. Quantity $v^0(c_\tau, t_\tau)$ denotes the null preference weight, which depends on the assortments offered by competitors. Then request τ chooses alternative $j \in S_\tau$ with probability

$$P_{j:S_\tau} = \frac{v(\boldsymbol{x}^{\tau,j})}{v^0(c_\tau, \ell_\tau) + \sum_{j' \in S_\tau} v(\boldsymbol{x}^{\tau,j'})}. \tag{2.7}$$

The parameters in (2.7) are estimated with the airline data using maximum likelihood estimation.

We calibrate an MNL model and a spiked-MNL model with the data of Monday flights in 2011, solve the CDLP problem under both models and derive the control heuristics accordingly, which specify the fractions of time offering different assortments. By default, we offer larger assortments first to each segment of customers. We also calibrate a spiked-MNL model with the data of Monday flights in 2012, and use this model in the simulation to evaluate the revenue performance of the controls derived under the MNL and the spiked-MNL models previously calibrated.

We consider the following performance metric. For a given booking control policy ψ, let $E[Z^{\psi}]$ denote the expected revenue achieved using policy ψ. Since the CDLP optimal value z^{CDLP} is an upper bound of the optimal expected revenue of optimization problem (2.1), we use the *revenue ratio* $\rho^{\psi} := \mathsf{E}[Z^{\psi}]/z^{\mathrm{CDLP}}$ as the performance metric of policy ψ. A good policy should yield a ratio ρ^{ψ} that is close to 1.

Table 2.1 shows the revenue ratio ρ^{ψ} averaged over 100 simulation runs with 95% confidence interval under the MNL and the spiked-MNL models. We see that the average revenue under the spiked-MNL model surpasses that under the MNL model by 5-7%.

Table 2.1: Average revenue ratio under the MNL and spiked-MNL models

model	avg. revenue ratio to CDLP	95% CI
MNL	0.891	[0.888, 0.895]
spiked-MNL	0.954	[0.952, 0.957]

2.8.2 Comparison between Different Booking Control Policies

In this subsection, we examine the performance of the booking control policies using the same airline data described previously. Again, we calibrate demand models with the data of Monday flights in 2011, using the spiked-MNL model. Then we use the calibrated demand models and calculate the booking control policies. We next calibrate demand models with the data of Monday flights in 2012, and use these demand models in the simulation to evaluate the performance of the booking control policies. We use the revenue ratio $\rho^{\psi} = \mathsf{E}[Z^{\psi}]/z^{\mathrm{CDLP}}$ as the performance metric, where $E[Z^{\psi}]$ denote the expected revenue achieved using policy ψ. A good policy should yield a ratio ρ^{ψ} close to 1.

We test the following booking control policies.

- **EMSR-b**: The nested booking limit heuristic proposed by [38], which is a popular heuristic used in airline reservation systems.
- **SBLP**: The nested booking limit heuristic proposed in Section 2.7, where the booking limits are constructed from the optimal solution of the SBLP.
- **CDLP**: This policy offers different assortments to different segments of customers over specified fractions of time, with fractions specified by the optimal solution of the CDLP. By default, we offer larger assortments first to each customer segment.

Note that SBLP and EMSR-b are all nested booking limit policies. There are two variants of nested booking limit policies, i.e., standard nesting and theft nesting. We implement both variants on the booking limit heuristics and use “-s” and “-t” to distinguish them. A detailed discussion on standard versus theft nesting can be found in [39] and [40].

Revenue performance of different policies. Figure 2.6 shows the revenue ratios ρ^{ψ} of the policies, with 95% confidence intervals, over 100 simulation runs. The CDLP-based heuristic has the best average performance among all the policies tested. However, we note that the CDLP-based heuristic specifies the fractions of time offering each assortment to each customer segment, and it is not directly implementable in current airline IT systems. The SBLP-based booking limit heuristics perform slightly worse, but they outperform the EMSR-b heuristics currently used in the airline industry, capturing an additional 2-4% revenue.

Robustness of different policies. Next, we tweak the model parameters and show the robustness of different policies. We follow the strategy proposed in [17] and evaluate the policies under different load factors. Specifically, we scale the capacity by a factor $k_1 \in \{0.8, 1.0, 1.2\}$ and the null preference weight by a factor $k_2 \in \{0.8, 1.0, 1.2\}$. We

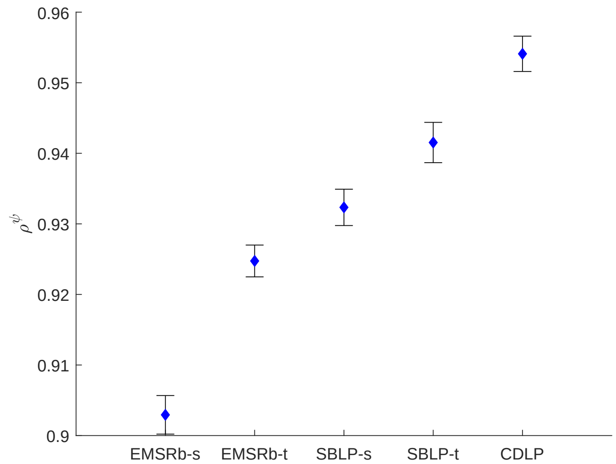

Figure 2.6: Revenue ratios over a real-world dataset under different policies

Table 2.2: Revenue ratios under different capacity and null preference weight scaling

k_1	k_2	EMSRb-s	EMSRb-t	SBLP-s	SBLP-t	CDLP
0.8	0.8	0.929	0.941	0.928	0.939	0.888
	1.0	0.911	0.925	0.920	0.929	0.916
	1.2	0.894	0.915	0.916	0.924	0.930
1.0	0.8	0.909	0.932	0.929	0.943	0.955
	1.0	0.903	0.925	0.932	0.942	0.956
	1.2	0.905	0.921	0.934	0.938	0.958
1.2	0.8	0.882	0.911	0.909	0.929	0.961
	1.0	0.886	0.907	0.911	0.924	0.965
	1.2	0.895	0.907	0.917	0.921	0.967

report in Table 2.2 the revenue ratios ρ^{ψ} under these scalings, averaged over 100 simulation runs. We observe that although on average the CDLP-based heuristic performs well, there are cases where it fails to beat the booking limit policies. The revenue performance under the booking limit policies, the EMSR-b and the SBLP-based heuristics, is robust against the scaling of capacity and null preference weight. And SBLP-based heuristics achieve up to 5% gains in revenue over the EMSR-b heuristics.

2.9 Concluding Remarks

In this chapter, we consider a network revenue management problem under the spiked-MNL choice model. The spiked-MNL model is a variant of the MNL model and is motivated by the phenomenon of the spike effect observed by airlines. We show that efficient sets under the spiked-MNL model are nested-by-revenue assortments. We also consider deterministic approximations under the spiked-MNL model and propose static booking limit heuristics, which are shown to be near-optimal.

Future research can explore other forms of spike effects along dimensions other than price. Considering pricing problems under the spiked-MNL model is another interesting research avenue to follow.

CHAPTER 3
DYNAMIC PRICING FOR TRUCKLOAD TRANSPORTATION MARKETPLACES

3.1 Overview

Truck transportation is an important part of supply chains and the overall economy. According to the American Trucking Associations (ATA), the US trucking industry generated a total of $796.7 billion revenue in 2018. The truck transportation market can be classified into parcel transportation, private trucking, less-than-truckload transportation, and for-hire truckload transportation. Of these, for-hire truckload transportation has a share of about 50% of all trucking revenues in the US. Also, truckload transportation exhibits much smaller economies of scale than parcel and less-than-truckload transportation. Therefore, whereas the parcel and less-than-truckload transportation markets are concentrated with a few large carriers, the for-hire truckload transportation market is fragmented with thousands of small carriers, many of which operate only a few trucks or are owner-operators (i.e., individual drivers with a single truck). As of May 2019, the number of for-hire truckload carriers on file with the US Federal Motor Carrier Safety Administration totaled 892,078, with 91.3% operating less than six trucks [41]. This fragmentation creates an opportunity for market makers in the for-hire truckload transportation market, who serve as intermediaries between carriers and shippers.

Traditionally, truckload brokers and freight forwarders have been such market makers, and they performed their functions in a labor-intensive manner, involving time-consuming relationship building and communication. Information technology facilitates the possibility of automating some of the labor-intensive market-making activities, and the prevalence of mobile telecommunication enables even traveling drivers to interact directly with market makers. Therefore, it is not surprising that both traditional truckload brokers and new tech companies such as Uber Freight, Amazon Freight, and Convoy are building online truckload marketplaces that streamline the matching process between the demand side (shippers) and the supply side (carriers) and automate activities such as price quotation and booking.

While known as "Uber for trucks," the pricing model of for-hire truckload marketplaces differs from that of ride-hailing marketplaces (e.g., Uber and Lyft) in two major aspects. First, ride-hailing marketplaces usually offer dynamic prices to both riders and drivers in real time. In contrast, it is typical for a truckload market maker to negotiate contracts with shippers that specify prices (or more generally price formulas) for truckload shipments, also known as *loads*. Each contract remains in effect for a long period of time, usually a year, and covers all loads offered by a shipper during that time. When the marketplace receives a load transportation request from a shipper, the market maker chooses a price to offer to carriers and publishes the load data as well as the offered price on the marketplace. The load data includes the pickup location, delivery location, cargo classification, and scheduled pickup time (or time window, usually not more than a few hours in length — here we treat the start of the time window as the pickup time). Under the contract, the market maker is responsible for acquiring transportation service for the load before its scheduled pickup time. The market maker may have to increase the load's offered price if it cannot find a carrier willing to accept the offer — sometimes, if supply is small relative to demand, this price may even go above the contract price that the shipper pays. Clearly, this business model exposes the market maker to risks associated with carrier supply uncertainty.

Second, ride-hailing marketplaces often determine how to match drivers and riders and do not allow drivers to choose among the riders in the marketplace. In contrast, truckload marketplaces allow carriers to choose among the loads in the marketplace. This is because it is more important for a truckload carrier than for a ride-hailing driver to evaluate many factors before choosing a load, such as the load's origin and destination, pickup time, cargo classification, deadhead miles (empty miles from the delivery location of the truck's pre-

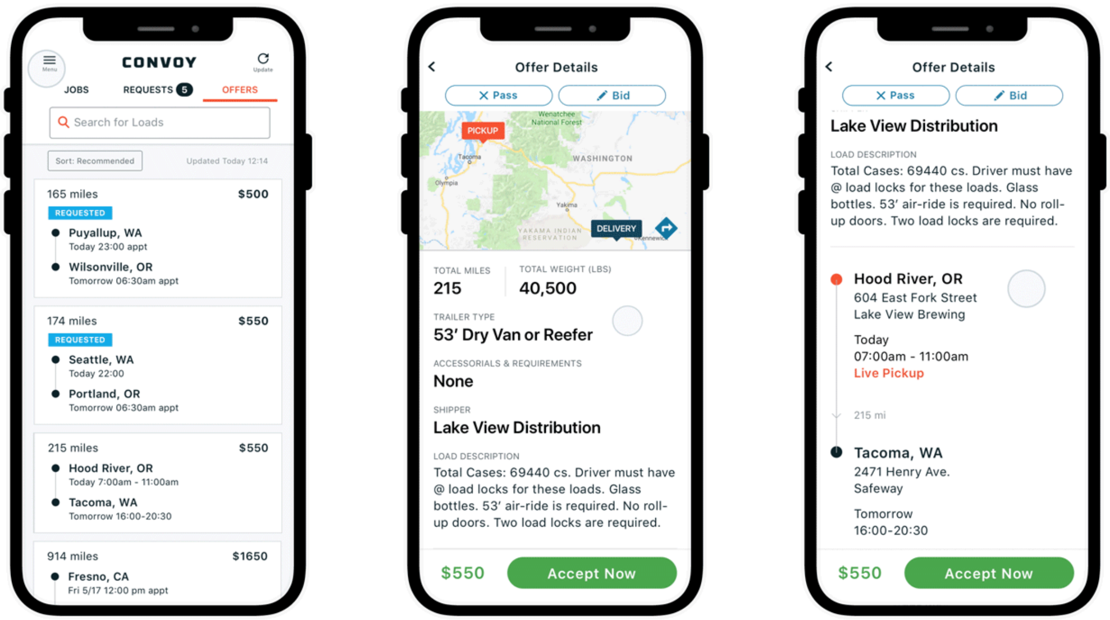

Figure 3.1: A truckload marketplace app. (source: convoy.com)

vious load or the truck's current location to the pickup location of the new load), etc. For example, the truck driver may not be able to pick up a load at the scheduled pickup time due to the truck's current location or due to hours of service regulations. Also, a truck has a specific type of trailer that can handle only some cargo classes, and the truck driver often has preferences regarding pickup and delivery locations, shippers, cargo types, and schedule. Therefore it is more practical to let carriers choose among loads than for the market maker to determine the matching of trucks and loads. Although the market maker does not determine the assignment of trucks to loads, the market maker influences the choices of carriers through pricing.

To illustrate a carrier's interaction with a marketplace, Figure 3.1 shows some mobile app screens of a typical truckload marketplace. The first screen shows a list of loads currently available on the marketplace among which a carrier can choose, with data such as price, origin and destination, pickup and drop-off time. The list of loads can be filtered by attributes such as their pickup time relative to the current time, which can range from a few hours to several weeks. The second and third screens show more details of a specific load, including the type of trailer needed, total load weight, and total distance between the pickup and delivery locations. A carrier can choose and book a load immediately. Since carriers can choose among different loads, it is important for the market maker to take into account carriers' choice behavior when it sets prices. Nevertheless, we noticed that some pricing models used by these market makers consider each load separately, without accounting for carriers' choices among the available loads [42, 43].

This chapter considers a dynamic pricing problem for such for-hire truckload transportation marketplaces, where we incorporate carriers' choice behavior among available loads. In the model both loads and carriers arrive randomly to the marketplace. Each load has a deadline, and the market maker wants to get a carrier to accept the load before its deadline. If a load expires, i.e., it is not accepted by a carrier before its deadline, the market maker incurs a penalty. The time until a load expires is called the load's *lead time*. The market maker offers a price for each load currently available, where the price of a load can vary based on its lead time and the market conditions, e.g., how many loads are available in total. Carriers arrive to the marketplace and view the available load data such as price, origin-destination locations, cargo class, schedule, etc. Then the carrier makes a booking choice or leaves without accepting any load. The market maker can dynamically adjust the price for each load to minimize the expected cost, including the price paid to carriers and penalty costs incurred by expiring loads.

In most of the chapter, we assume that loads are partitioned into types based on load data other than the loads' lead times, and that there is a separate marketplace for each type of load. For example, loads can be partitioned based on the origin-destination zones and the type of trailer needed. With such a partitioning, one load type can include all loads that need a dry van (an enclosed truck trailer) on a certain shipping lane (e.g., from Atlanta to New York). From the point of view of the market maker, the two major factors that distinguish loads of the same type are their prices and lead times. A carrier may also take into account other load data that distinguish loads of the same type, such as the identity of the shipper or the exact pickup and delivery locations. Thus, the market maker uses a choice model in which the load attributes are lead time and price, but carriers may have preferences among loads with the same lead time and price. This basic model captures the main effect of lead time on the pricing of loads. In the extension section, we discuss how to extend the basic model to the setting in which carriers choose among multiple types of loads, e.g., dry van shipments on multiple shipping lanes to different destinations.

Main Contributions: We formulate the market maker's dynamic pricing problem as a discrete-time, infinite horizon, average cost Markov decision process (MDP). Unlike previous models of dynamic pricing for a single load [42, 43], we consider a model in which carriers choose among available loads of the same type. More specifically, the problem includes a multinomial logit choice model, in which each available load is an alternative, and the preference weight of each load is determined by its lead time and its price.

Optimal Policy Structure. We show that the optimal price in each state of the MDP can be computed by solving a convex optimization problem, given the marginal cost of each load. Thus, when using iterative methods, such as relative value iteration, to solve the MDP, in each iteration the optimization problem for each state can be solved efficiently. Regarding the structure of the optimal pricing policy, we show that the optimal prices are higher for loads with shorter lead times (Theorem 3.3).

Discrete-Time Fluid Model and Asymptotically Optimal Policy. The MDP has a state space that increases exponentially with the number of lead times. Therefore we consider an approximation of the MDP, which we call the *discrete-time fluid model* (DTFM). We propose a tractable reformulation of the DTFM, and we show that its optimal objective value provides a lower bound on the long-run average cost of the MDP under any stationary policy (Theorem 3.4). Based on the DTFM, we propose a simple pricing policy for the MDP that is easy to compute. The pricing policy specifies a price for each lead time that does not vary over time. Even though the price for each lead time is time invariant, the price of a load still varies as it comes closer to its deadline. We show that the proposed policy is asymptotically optimal with a loss ratio of order $O(1/\theta)$, where θ represents the scale of shipper demand and carrier supply (Theorem 3.8). In contrast, in the existing dynamic pricing literature, it is more typical to obtain loss ratios of order $O(1/\sqrt{\theta})$ from fluid approximations (e.g. [44]). We provide some intuition to explain this distinction.

Continuous-Time Fluid Model. We also present a *continuous-time fluid model* (CTFM) as an approximation of the MDP. We show that the CTFM can be reformulated with a convex objective and linear constraints. We also examine the structure of the optimal solution of the CTFM (Theorem 3.12). We show that, for an arbitrary load arrival pattern and general carrier preferences, the optimal prices always increase as lead time decreases. We discuss extensions of the CTFM to incorporate various practical considerations.

3.2 Related Literature

Many papers have studied dynamic pricing for perishable products (e.g., seasonal goods, airline seats) under choice models. The most widely used choice model in revenue management is the multinomial logit choice model. For example, [21] and [45] studied dynamic pricing problems for substitute products under the multinomial logit model and established structural properties for the optimal pricing policy. Reference [46] used the multinomial logit model to study a dynamic pricing problem with a horizontally differentiated assortment and show that individual product availability drives pricing. The authors of [47] studied a dynamic pricing problem in which different customer segments have disjoint consideration sets, and customers in each segment choose from its consideration set according to a multinomial logit model. Dynamic pricing problems with other types of choice models, such as the nested logit model [48, 26] and generalized extreme value models [49], have also been studied. In most of these papers, sales of the perishable products start at the same time and end at the same time, so the system state simply represents the current inventory level of each product. In truckload transportation pricing, an interesting feature is that at any point in time different loads have different deadlines, which adds another dimension to the system state.

Another related stream of research is inventory management of perishable products when the demands are differentiated based on the freshness of products. For example, such problems arise in the management of blood products such as blood platelets, where different levels of blood freshness are required for different medical procedures [50, 51].

Although many dynamic pricing problems can be formulated as MDPs, these formulations are often computationally intractable because the number of dimensions of the state space is large, usually because the number of products is large. A popular family of heuristics uses deterministic fluid approximations of the MDP, in which random variables are replaced by their means. For example, [44] and [52] considered fluid approximations of finite horizon dynamic pricing problems, and solved the fluid problems to obtain prices for static-pricing heuristics. The authors of [44] showed that the resulting heuristics are asymptotically optimal with a total loss of order $O(\sqrt{\theta})$ when both customer demand and inventory supply are scaled by a factor of θ. Similar asymptotic results have been established for revenue management problems with choice models [17]. The $O(\sqrt{\theta})$ loss rate is tight for any static policy; in order to improve this rate, the fluid problem can be re-optimized during the time horizon to obtain a more adaptive pricing policy [53, 54, 55, 56]. Reference [57] proposes a policy that requires a single optimization at the beginning of the selling horizon and that adjusts the prices of specified products according to a price adjustment that is linear in the observed demand. Our approach to truckload transportation pricing is related to this line of research. We consider a fluid approximation of the dynamic pricing problem and use the optimal solution of the fluid problem to construct a pricing policy with a time invariant price for each lead time. We show that this policy achieves $O(1/\theta)$ loss ratio, as opposed to the usual $O(1/\sqrt{\theta})$ loss ratio.

Research about marketplaces from the point of view of market makers has increased recently, especially motivated by the ride-hailing industry. Pricing problems for ride-hailing marketplaces and compared static and dynamic pricing policies have been studied [58, 59, 60, 61, 62, 63, 64]. For example, [59] proposed a state independent pricing policy in a closed queueing network and showed that the policy approaches the optimal one as the number of vehicles in the system grows. Researchers have also considered matching problems for ride-hailing marketplaces and investigated the asymptotic performance of several policies [65, 66]. The problem of joint matching and pricing in ride-hailing marketplaces has been studied in [67], and the authors argued that joint optimization leads to a significant performance improvement.

For truckload transportation marketplaces, [43] and [42] considered dynamic carrier pricing with application to Uber Freight and Amazon Freight, respectively. However, both of these works considered pricing of each load separately without accounting for carriers' choice behavior, whereas this chapter considers dynamic pricing for all available loads in the marketplace taking carriers' choice behavior into account with a discrete choice model. Another stream of research considered truckload pricing and load assignment from a carrier's perspective. For example, [68, 69], and [70] proposed models to optimize carrier pricing and load assignment.

3.3 Problem Formulation

We consider a truckload transportation marketplace with a market maker that sells truckload transportation to shippers and buys truckload transportation from carriers. Contracts between the market maker and the shipper that specify the prices that shippers pay to the market maker for transportation services have already been established. Shippers submit load transportation requests to the marketplace at random times. Each load has a scheduled pickup time, also called the load's deadline, and different loads enter the marketplace with different amounts of time to go until the load's deadline. For each load in the marketplace the market maker sets a price that is offered to carriers for transporting the load, and the market maker can adjust these prices dynamically. Carriers check the available loads in the marketplace and their attributes, including prices, at random times. A carrier can choose to accept a load, in which case the load is booked for transportation by the carrier, the load is removed from the marketplace, and the market maker pays the carrier the price that was offered at the time of the booking. A carrier can also choose not to accept any load in the marketplace. If a load reaches its deadline without being booked by a carrier, then we say that the load expires, the load is removed from the marketplace, and the market maker pays a penalty. We consider the problem from the point of view of the market maker. The market maker wants to determine a policy for offering prices to carriers that maximizes the market maker's long-run average profit per unit time. Since the prices that shippers pay to the market maker are fixed by contracts, and loads enter the marketplace according to an exogenous process, the objective is equivalent to minimizing the market maker's long-run average cost per unit time.

We partition time into discrete periods with sufficiently small length, such that in each period at most one new load enters the marketplace and at most one carrier arrives to check the available loads in the marketplace. Then we formulate the dynamic pricing problem as a discrete-time, infinite horizon, average cost Markov decision process (MDP). In each period, the following sequence of events occurs: (1) either one new load enters the marketplace with probability $\lambda \in (0, 1)$, or no load enters the marketplace with probability $1 - \lambda$; (2) the market maker sets a price for each load in the marketplace, excluding the expiring loads; (3) either one carrier arrives to check the available loads in the marketplace with probability $\mu \in (0, 1)$, and the carrier chooses which load to book or the carrier leaves without booking a load, or no carrier arrives with probability $1 - \mu$; (4) the market maker pays the carrier according to the offered prices if a carrier booked a load, as well as the penalty cost C for each expiring load.

The number of periods until a load expires is called the *(discrete) lead time* of the load. New loads can have different initial lead times. The maximum initial lead time is L. A new load has initial lead time $\ell \in \{1, \ldots, L\}$ with probability ψ_ℓ; we let $\psi_0 := 0$ for notation consistency. Therefore, the lead time of any load can take value $\ell \in \{0, \ldots, L\}$, where

$\ell = 0$ corresponds to the case that the load expires.

In a period, after (1) any new load has entered the marketplace, and before (2) the market maker has set a price for each load in the marketplace, let $x_\ell \in \mathbb{N}$ denote the number of loads with lead time ℓ in the marketplace. Let $\boldsymbol{x} := (x_\ell, \ell = 0, \ldots, L)$ denote the state vector, and let $\mathrm{supp}(\boldsymbol{x}) := \{\ell \geq 1 : x_\ell > 0\}$ denote the lead times $\ell \geq 1$ with a positive number of loads. Let p_ℓ denote the price that the market maker offers for each load with lead time ℓ. Let $\boldsymbol{p} := (p_\ell, \ell = 1, \ldots, L)$ denote the corresponding price vector. Let $\mathcal{P} \subset \mathbb{R}^L$ denote the set of feasible price vectors. Of course, only loads with lead times $\ell \in \mathrm{supp}(\boldsymbol{x})$ can be booked, so the prices p_ℓ for $\ell \notin \mathrm{supp}(\boldsymbol{x})$ will not affect carrier bookings or market maker costs. Also, the quantity x_0 represents the number of loads that expire in this period, which cannot be booked, so the market maker does not specify a price for lead time $\ell = 0$.

A carrier who checks the available loads in the marketplace chooses to book a load with lead time ℓ with probability $f_\ell(\boldsymbol{p}, \boldsymbol{x})$ for $\ell = 1, \ldots, L$. If $x_\ell = 0$, then $f_\ell(\boldsymbol{p}, \boldsymbol{x}) = 0$. Thus a carrier chooses not to book a load with probability $f_0(\boldsymbol{p}, \boldsymbol{x}) = 1 - \sum_{\ell=1}^{L} f_\ell(\boldsymbol{p}, \boldsymbol{x})$.

For $\ell = 0, \ldots, L$, let $\boldsymbol{e}^\ell \in \{0,1\}^{L+1}$ denote a vector with the ℓ-th entry being 1 and all other entries being 0, and let $\boldsymbol{0} \in \{0,1\}^{L+1}$ denote a vector with all 0 entries. Let random variable $A_\ell(t) = 1$ if a load enters the marketplace with initial lead time ℓ in period t, and let $A_\ell(t) = 0$ otherwise. Note that $A_0(t) = 0$ and $\sum_{\ell=1}^{L} A_\ell(t) \leq 1$. Let $\boldsymbol{A}(t) := (A_\ell(t), \ell = 0, \ldots, L)$ denote the random load arrival vector in period t. For $\ell = 0, \ldots, L$, let $X_\ell(t) \in \mathbb{N}$ denote the random number of loads with lead time ℓ in period t after any new load enters the marketplace. Let $\boldsymbol{X}(t) := (X_\ell(t), \ell = 0, \ldots, L)$ denote the random state vector in period t. Let $\boldsymbol{p}(t) \in \mathcal{P}$ denote the random price vector set by the market maker in period t. Let random variable $M_\ell(t) = 1$ if a carrier books a load with lead time ℓ in period t, and let $M_\ell(t) = 0$ otherwise. Note that $M_0(t) = 0$ and $\sum_{\ell=1}^{L} M_\ell(t) \leq 1$. Let $\boldsymbol{M}(t) := (M_\ell(t), \ell = 0, \ldots, L)$ denote the random load booking vector in period t. We assume that $\{\boldsymbol{A}(t)\}$ are independent and identically distributed. Also, $\boldsymbol{A}(t)$ is independent of the history $\mathcal{H}(t -$

$1) := \{(\boldsymbol{A}(\tau), \boldsymbol{X}(\tau), \boldsymbol{p}(\tau), \boldsymbol{M}(\tau)), \tau = 0, \ldots, t-1\}$ of the process. Specifically, given any history $\mathcal{H}(t-1)$ of the process, the load arrival probabilities are $\mathsf{P}[\boldsymbol{A}(t) = \boldsymbol{e}^\ell \,|\, \mathcal{H}(t-1)] = \lambda\psi_\ell$ for $\ell = 0, \ldots, L$, and $\mathsf{P}[\boldsymbol{A}(t) = \boldsymbol{0} \,|\, \mathcal{H}(t-1)] = 1 - \lambda$. Furthermore, given any history $\mathcal{H}(t-1)$ of the process, that the load arrival in period t is $\boldsymbol{A}(t)$, that the available loads in the market place is $\boldsymbol{X}(t)$, that the market maker set prices $\boldsymbol{p}(t)$ for period t, and that a carrier arrived in period t to check the available loads in the marketplace, the driver choice probabilities are $\mathsf{P}[\boldsymbol{M}(t) = \boldsymbol{e}^\ell \,|\, \mathcal{H}(t-1), \boldsymbol{A}(t), \boldsymbol{X}(t), \boldsymbol{p}(t)] = f_\ell(\boldsymbol{p}(t), \boldsymbol{X}(t))$ for $\ell = 1, \ldots, L$, and $\mathsf{P}[\boldsymbol{M}(t) = \boldsymbol{0} \,|\, \mathcal{H}(t-1), \boldsymbol{A}(t), \boldsymbol{X}(t), \boldsymbol{p}(t)] = f_0(\boldsymbol{p}(t), \boldsymbol{X}(t))$.

As time passes, the lead time of each load decreases, e.g., a load with lead time ℓ in the current period will have a lead time $\ell - 1$ in the next period. Let the left-shift operator $\mathcal{S}$ be given by

$$\mathcal{S}_\ell(\boldsymbol{x}) \;:=\; \begin{cases} x_{\ell+1}, & \ell = 0, \ldots, L-1, \\ 0, & \ell = L. \end{cases}$$

Then, the system dynamics is given by

$$\boldsymbol{X}(t+1) \;=\; \mathcal{S}\left(\boldsymbol{X}(t) - \boldsymbol{M}(t)\right) + \boldsymbol{A}(t+1).$$

Figure 3.2 provides a graphical illustration of the dynamics.

Dynamic Program Formulation. We consider the dynamic pricing problem over an infinite horizon. Assume that the initial state is $\boldsymbol{X}(1) = \boldsymbol{0}$. We aim to find an optimal stationary pricing policy that minimizes the long-run average cost (payments to carriers plus expiration penalties) per period for the market maker. Given the current state $\boldsymbol{x}$, a stationary policy $\boldsymbol{\varphi} : \mathbb{N}^{L+1} \mapsto \mathcal{P}$ specifies the price vector $\boldsymbol{\varphi}(\boldsymbol{x}) = (\varphi_\ell(\boldsymbol{x}), \ell = 1, \ldots, L)$ to offer carriers in the marketplace. For any stationary policy $\boldsymbol{\varphi}$, let $\gamma^{\boldsymbol{\varphi}}$ denote the long-run

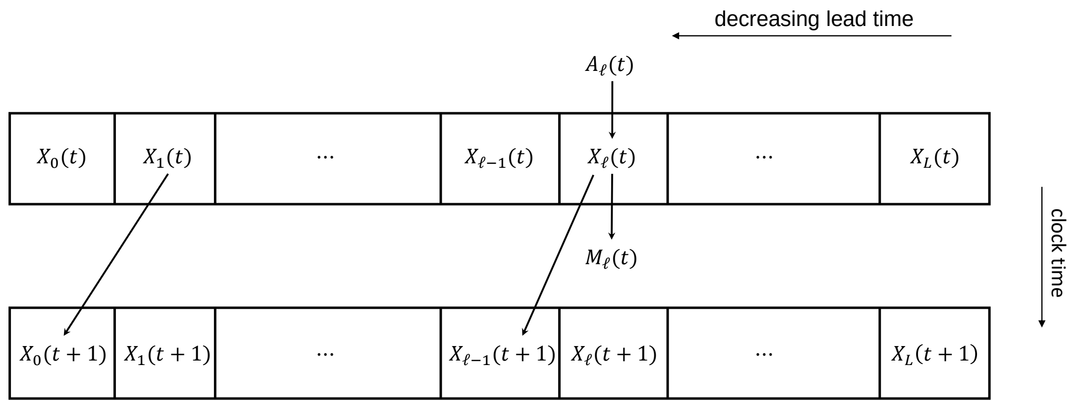

Figure 3.2: Illustration of the load dynamics in the MDP

average cost per period under policy $\boldsymbol{\varphi}$, given by

$$\gamma^{\boldsymbol{\varphi}} \;:=\; \lim_{T\to\infty} \frac{1}{T}\,\mathsf{E}\left[\sum_{t=1}^{T}\left(C\,X_0(t) + \sum_{\ell=1}^{L}\varphi_\ell(\boldsymbol{X}(t))\,M_\ell(t)\right)\,\middle|\, \boldsymbol{X}(1)=\boldsymbol{0}\right].$$

Remark 3.1. *A sufficient condition for the long-run average cost per period $\gamma^{\boldsymbol{\varphi}}$ to not depend on the initial state is the existence of a special state such that, for all initial states and all policies, the special state is recurrent (see, e.g., Theorem 5.5.1 in [71]). For the system dynamics specified above, the state $\boldsymbol{x} = \boldsymbol{0}$ is such a special state because $\lambda < 1$, since starting from any initial state there is a positive probability $(1-\lambda)^{L+1}$ that no new load arrives in $L+1$ consecutive periods and the system state becomes $\boldsymbol{x} = \boldsymbol{0}$. Therefore the assumption that the initial state is $\boldsymbol{X}(1) = \boldsymbol{0}$ is without loss of generality.*

Let $\gamma^* := \inf_{\boldsymbol{\varphi}} \gamma^{\boldsymbol{\varphi}}$ denote the optimal long-run average cost per period, and let $h^*(\boldsymbol{x})$ denote the optimal differential cost in state $\boldsymbol{x}$. They satisfy the following optimality equation:

$$\begin{aligned}
&\gamma^* + h^*(\boldsymbol{x}) \\
&\overset{(a)}{=} \inf_{\boldsymbol{p}\in\mathcal{P}} \Bigg\{ C\,x_0 \\
&\qquad\qquad + \mu \sum_{\ell\in\operatorname{supp}(\boldsymbol{x})} f_\ell(\boldsymbol{p},\boldsymbol{x}) \left[p_\ell + \lambda\sum_{k=1}^{L}\psi_k h^*\left(\mathcal{S}\left(\boldsymbol{x}-\boldsymbol{e}^\ell\right)+\boldsymbol{e}^k\right) + (1-\lambda)h^*\left(\mathcal{S}\left(\boldsymbol{x}-\boldsymbol{e}^\ell\right)\right)\right]
\end{aligned}$$

$$
\begin{aligned}
&\quad + \left[1 - \mu \sum_{\ell \in \text{supp}(\boldsymbol{x})} f_\ell(\boldsymbol{p}, \boldsymbol{x})\right] \left[\lambda \sum_{k=1}^{L} \psi_k h^*\left(\mathcal{S}(\boldsymbol{x}) + \boldsymbol{e}^k\right) + (1-\lambda) h^*\left(\mathcal{S}(\boldsymbol{x})\right)\right] \Bigg\} \\
&\overset{(b)}{=} \inf_{\boldsymbol{p} \in \mathcal{P}} \Bigg\{ C\, x_0 + \mu \sum_{\ell \in \text{supp}(\boldsymbol{x})} f_\ell(\boldsymbol{p}, \boldsymbol{x}) \left[p_\ell - \Delta_\ell(\boldsymbol{x})\right] \\
&\quad + \lambda \sum_{k=1}^{L} \psi_k h^*\left(\mathcal{S}(\boldsymbol{x}) + \boldsymbol{e}^k\right) + (1-\lambda) h^*\left(\mathcal{S}(\boldsymbol{x})\right) \Bigg\} \quad \text{for all } \boldsymbol{x} \in \mathbb{N}^{L+1}, \qquad \text{(MDP)}
\end{aligned}
$$

where terms on the right side of equality (a) represent the penalty cost incurred by expired loads, the expected cost if a carrier makes a booking, and the expected cost if no load is booked, and step (b) simplifies (a) by defining the *displacement cost* of a load with lead time ℓ as

$$
\begin{aligned}
\Delta_\ell(\boldsymbol{x}) \;:=\; & \lambda \sum_{k=1}^{L} \psi_k \left[h^*\left(\mathcal{S}(\boldsymbol{x}) + \boldsymbol{e}^k\right) - h^*\left(\mathcal{S}\left(\boldsymbol{x} - \boldsymbol{e}^\ell\right) + \boldsymbol{e}^k\right)\right] \\
& + (1-\lambda)\left[h^*\left(\mathcal{S}(\boldsymbol{x})\right) - h^*\left(\mathcal{S}\left(\boldsymbol{x} - \boldsymbol{e}^\ell\right)\right)\right] \qquad \text{for } \ell \in \text{supp}(\boldsymbol{x}).
\end{aligned}
$$

Intuitively, the displacement cost $\Delta_\ell(\boldsymbol{x})$ represents the marginal cost in state $\boldsymbol{x}$ for an extra load with lead time ℓ.

Also, let

$$
\begin{aligned}
G(\boldsymbol{p}, \boldsymbol{x}) \;&:=\; \mu \sum_{\ell \in \text{supp}(\boldsymbol{x})} f_\ell(\boldsymbol{p}, \boldsymbol{x}) \left[p_\ell - \Delta_\ell(\boldsymbol{x})\right], \\
H(\boldsymbol{x}) \;&:=\; C\, x_0 + \lambda \sum_{k=1}^{L} \psi_k h^*\left(\mathcal{S}(\boldsymbol{x}) + \boldsymbol{e}^k\right) + (1-\lambda) h^*\left(\mathcal{S}(\boldsymbol{x})\right).
\end{aligned}
$$

Then the optimality equation can be written concisely as

$$
\gamma^* + h^*(\boldsymbol{x}) \;=\; \inf_{\boldsymbol{p} \in \mathcal{P}} G(\boldsymbol{p}, \boldsymbol{x}) + H(\boldsymbol{x}) \qquad \text{for all } \boldsymbol{x} \in \mathbb{N}^{L+1}.
$$

Since $H(\boldsymbol{x})$ does not depend on $\boldsymbol{p}$, it suffices to optimize $G(\boldsymbol{p}, \boldsymbol{x})$ to get an optimal price vector for state $\boldsymbol{x}$.

3.4 Optimal Dynamic Pricing under the Multinomial Logit Model

Next, we consider the dynamic pricing problem under the multinomial logit model. First, we specify the multinomial logit model used by the market maker. Then we show how to derive the optimal price vector as a function of the displacement costs. Finally, we establish some structural properties of the optimal pricing policy.

3.4.1 Multinomial Logit Model

The market maker uses the following multinomial logit model of carriers' choice behavior among available loads. Given a price p, a load with lead time ℓ has a preference weight

$$v_\ell(p) \;:=\; \exp(\beta p + \beta_\ell^0), \qquad \text{(Heterogeneous Preference)}$$

where $\beta > 0$ denotes the price sensitivity coefficient, and β_ℓ^0 represents a carrier's mean non-monetary utility for a load with lead time ℓ. The parameter β_ℓ^0 is allowed to depend on the lead time ℓ. This is called the Heterogeneous Preference setting. For example, everything else being equal, carriers may prefer loads with closer pickup times (i.e., smaller lead times), in which case β_ℓ^0 is decreasing in ℓ.

In the special case in which a carrier's mean non-monetary utility depends only on price but not on lead time, we denote $\beta_\ell^0 = \beta_0$ for all ℓ. This is called the Homogeneous Preference setting. In this case, given a price p, a load with lead time ℓ has a preference weight

$$v_\ell(p) \;=\; \exp(\beta p + \beta_0). \qquad \text{(Homogeneous Preference)}$$

A carrier may choose not to book a load in the marketplace. The preference weights are scaled so that the preference weight of this null alternative is normalized to 1. Then, according to the multinomial logit model [1, 2], given any state $\boldsymbol{x} \in \mathbb{N}^{L+1}$ and price vector

$\boldsymbol{p} \in \mathcal{P}$, the probability that a carrier books a load with lead time ℓ is given by

$$f_\ell(\boldsymbol{p}, \boldsymbol{x}) \;=\; \frac{x_\ell \; v_\ell(p_\ell)}{\sum_{k=1}^{L} x_k \; v_k(p_k) + 1},$$

and the probability that a carrier choose not to book a load in the marketplace is given by

$$f_0(\boldsymbol{p}, \boldsymbol{x}) \;=\; \frac{1}{\sum_{k=1}^{L} x_k \; v_k(p_k) + 1}.$$

Note that if $x_\ell = 0$ for some ℓ, then the choice probability $f_\ell(\boldsymbol{p}, \boldsymbol{x}) = 0$, meaning that a carrier cannot choose a load with lead time $\ell \notin \text{supp}(\boldsymbol{x})$, since no such load is available in the marketplace.

3.4.2 Deriving the Optimal Price in Each State

Next we derive an expression for the optimal price vector. Given state $\boldsymbol{x}$ and displacement cost $\boldsymbol{\Delta}(\boldsymbol{x}) := (\Delta_\ell(\boldsymbol{x}), \ell \in \text{supp}(\boldsymbol{x}))$, the optimal price vector is the solution of the problem

$$\min_{\boldsymbol{p} \in \mathcal{P}} \quad G(\boldsymbol{p}, \boldsymbol{x}). \tag{3.1}$$

In the discussion below, the state $\boldsymbol{x}$ is fixed, so we write the displacement cost as $\boldsymbol{\Delta} = \boldsymbol{\Delta}(\boldsymbol{x})$ to simplify notation.

The objective function of (3.1) is non-convex, therefore we introduce the following change of variables. Let

$$u_\ell \;=\; \frac{x_\ell \; v_\ell(p_\ell)}{\sum_{k \in \text{supp}(\boldsymbol{x})} x_k \; v_k(p_k) + 1} \quad \text{and} \quad u_0 \;=\; \frac{1}{\sum_{k \in \text{supp}(\boldsymbol{x})} x_k \; v_k(p_k) + 1}, \tag{3.2}$$

and let $\boldsymbol{u} := (u_\ell, \ell = 1, \ldots, L)$. Note that

$$\sum_{\ell \in \text{supp}(\boldsymbol{x})} u_\ell + u_0 \;=\; 1 \tag{3.3}$$

and

$$\frac{u_\ell}{u_0} \;=\; x_\ell v_\ell(p_\ell) \;=\; x_\ell \exp(\beta p_\ell + \beta_\ell^0) \quad \Rightarrow \quad p_\ell \;=\; \frac{1}{\beta}\left(\ln\left(\frac{u_\ell}{x_\ell u_0}\right) - \beta_\ell^0\right). \tag{3.4}$$

Thus the objective function of (3.1) is equal to

$$\begin{aligned}\mu \sum_{\ell\in\text{supp}(\boldsymbol{x})} f_\ell(\boldsymbol{p},\boldsymbol{x})\,[p_\ell - \Delta_\ell] &\overset{(a)}{=} \frac{\mu}{\beta}\sum_{\ell\in\text{supp}(\boldsymbol{x})} u_\ell\left[\ln\left(\frac{u_\ell}{x_\ell u_0}\right) - \beta\Delta_\ell - \beta_\ell^0\right]\\ &\overset{(b)}{=} \frac{\mu}{\beta}\sum_{\ell\in\text{supp}(\boldsymbol{x})} u_\ell\left[\ln\left(\frac{u_\ell}{x_\ell}\right) - \beta\Delta_\ell - \beta_\ell^0\right] - \frac{\mu}{\beta}(1-u_0)\ln(u_0),\end{aligned}$$

where equality (a) follows from variable substitution (3.2) and (3.4), and equality (b) follows from arranging terms and (3.3). The right side of equality (b) is convex in $(\boldsymbol{u}, u_0)$.

Thus we consider the problem

$$\min_{(\boldsymbol{u},u_0)\in\mathbb{R}_+^{L+1}} \quad \frac{\mu}{\beta}\sum_{\ell\in\text{supp}(\boldsymbol{x})} u_\ell\left[\ln\left(\frac{u_\ell}{x_\ell}\right) - \beta\Delta_\ell - \beta_\ell^0\right] - \frac{\mu}{\beta}(1-u_0)\ln(u_0) \tag{3.5a}$$

$$\text{s.t.} \quad \sum_{\ell\in\text{supp}(\boldsymbol{x})} u_\ell + u_0 \;=\; 1 \tag{3.5b}$$

$$\frac{1}{\beta}\left(\ln\left(\frac{u_\ell}{x_\ell u_0}\right) - \beta_\ell^0\right) \;\in\; \mathcal{P} \qquad \text{for } \ell\in\text{supp}(\boldsymbol{x}). \tag{3.5c}$$

Given an optimal solution $(\boldsymbol{u}^*, u_0^*)$ for problem (3.5), one can compute an optimal price $\boldsymbol{p}^*$ for problem (3.1) as follows:

$$p_\ell^* \;=\; \frac{1}{\beta}\left(\ln\left(\frac{u_\ell^*}{x_\ell u_0^*}\right) - \beta_\ell^0\right) \qquad \text{for } \ell\in\text{supp}(\boldsymbol{x}).$$

For some feasible price sets $\mathcal{P}$, problem (3.5) is a convex optimization problem. For example, with lower and upper bounds on prices, that is, if $\mathcal{P} = [\underline{p}_1, \overline{p}_1]\times\cdots\times[\underline{p}_L, \overline{p}_L]$, then constraint (3.5c) reduces to

$$\exp(\beta\underline{p}_\ell + \beta_\ell^0)x_\ell u_0 \;\le\; u_\ell \;\le\; \exp(\beta\overline{p}_\ell + \beta_\ell^0)x_\ell u_0 \qquad \text{for } \ell\in\text{supp}(\boldsymbol{x}).$$

Then problem (3.5) has a convex objective and linear constraints. In Appendix B.6, we further study the structure of the optimal solution for problem (3.5) when $\mathcal{P} = \mathbb{R}_+^L$, and for some special cases we derive a closed-form expression for the optimal price as a function of $\boldsymbol{\Delta}$. In the remainder of the paper, we let all prices be unconstrained, that is, $\mathcal{P} = \mathbb{R}^L$. The motivation is that if the input of the optimization problem, including the arrival rates of loads and carriers and the carrier choice model, are accurate, then the optimization problem should produce good solutions without additional price constraints.

Remark 3.2. *When defining the optimal price vector $\boldsymbol{p}$ in the* MDP*, we implicitly assume that all loads with the same lead time $\ell \in \{1, \ldots, L\}$ must be offered at the same price, namely p_ℓ. This assumption is without loss of generality: It can be shown that under the multinomial logit model, it is never optimal to offer different prices for two loads with the same lead time. A proof is given in Appendix B.1.*

3.4.3 Structure of the Optimal Prices

Next we study the structure of the optimal prices under the multinomial logit model. Optimal prices exhibit the following monotonicity structure in the Homogeneous Preference setting: In any state, the optimal price is higher for loads with a shorter lead time.

Theorem 3.3. *In the* Homogeneous Preference *setting, in any state, the optimal price is higher for loads with a shorter lead time than for loads with a longer lead time. That is, for a given state $\boldsymbol{x}$, if $\boldsymbol{p}^*(\boldsymbol{x})$ is an optimal price vector, then $p_i^*(\boldsymbol{x}) \geq p_j^*(\boldsymbol{x})$ for any $i, j \in$ supp$(\boldsymbol{x})$ such that $i < j$.*

The proof is given in Appendix B.1. A consequence of this result is that if the market is thick and the system state varies little over time, then a load's price will tend to increase over time until it is booked. We consider this setting more formally in Section 3.7. The result above does not necessarily hold in the more general Heterogeneous Preference setting, in which carriers' mean non-monetary utility depends on the lead time. For example, loads with a shorter lead time may be more attractive to some carriers. In such a case, it may not be optimal to post higher prices for loads with shorter lead times; see Appendix B.7 for a counter-example.

3.5 Discrete-Time Fluid Approximation

In the formulation of the MDP, we choose the time periods small enough for the assumption that at most one load arrives and at most one carrier arrives in each time period to be accurate enough in a practical application. Thus, in an application the number of lead times L may be large. Unfortunately, the MDP is intractable when L is large, as the size of the state space is $\Omega(2^L)$. This motivates us to consider a discrete-time fluid approximation of the MDP. In this fluid approximation, the number of loads per lead time can be fractional, and we consider the system in steady state. That is, the fluid system state does not vary as loads are arriving and are being booked. The resulting fluid optimization problem is a single-stage optimization problem.

First, we formulate a discrete-time fluid model with price as the control. Then, we transform the problem into a convex optimization problem. We show that the optimal objective value of the fluid optimization problem is a lower bound for the long-run average cost per period for the MDP under any stationary policy. In Section 3.6, we propose a pricing policy for the MDP based on the optimal solution of the fluid optimization problem, and we establish performance guarantees for this pricing policy.

3.5.1 Discrete-Time Fluid Model

First we present the dynamics of the fluid model. In each period, the following sequence of events occurs: 1) new loads enter the marketplace, with $\lambda\psi_\ell$ new loads arriving for lead time ℓ, resulting in x_ℓ loads in the marketplace with lead time ℓ; 2) the market maker sets a price p_ℓ for each lead time ℓ; 3) carriers check the offers in the marketplace and book $\mu f_\ell(\boldsymbol{p}, \boldsymbol{x})$ loads for each lead time ℓ; 4) the market maker pays the carriers and incurs the penalty cost due to expiring loads, and all remaining loads in the system will have their lead times reduced by one. Figure 3.3 illustrates the dynamics of such a fluid system.

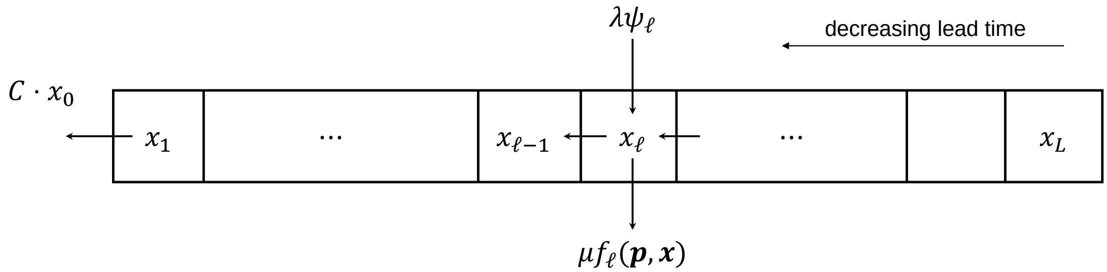

Figure 3.3: Illustration of the load dynamics in the discrete-time fluid model.

We consider the fluid system in a steady state; that is, the system state $\boldsymbol{x}$ and the price control $\boldsymbol{p}$ stay the same from period to period. This results in the following single-stage optimization problem, called the discrete-time fluid model (DTFM):

$$\hat{R} := \min_{\boldsymbol{x} \geq \boldsymbol{0},\, \boldsymbol{p}} \quad \mu \sum_{\ell=1}^{L} \frac{p_\ell \, x_\ell \, v_\ell(p_\ell)}{\sum_{k=1}^{L} x_k \, v_k(p_k) + 1} + C \, x_0 \tag{DTFM}$$

$$\text{s.t.} \quad x_\ell - x_{\ell-1} = \mu \frac{x_\ell \, v_\ell(p_\ell)}{\sum_{k=1}^{L} x_k \, v_k(p_k) + 1} - \lambda \psi_{\ell-1} \qquad \forall \ell \geq 1 \tag{3.6a}$$

$$x_L = \lambda \psi_L \tag{3.6b}$$

$$x_\ell \geq \mu \frac{x_\ell \, v_\ell(p_\ell)}{\sum_{k=1}^{L} x_k \, v_k(p_k) + 1} \qquad \forall \ell \geq 1 \tag{3.6c}$$

The objective of the DTFM minimizes the cost per period, where the first term represents the payment to carriers as they book loads, and the second term Cx_0 represents the penalty cost incurred due to expiring loads. Constraints (3.6a) and (3.6b) represent the system dynamics shown in Figure 3.3. The change in state during a time period is governed by two opposing forces: carriers booking available loads and new loads entering the marketplace. For lead time L, the system state (which is observed before carriers book loads during the time period) is solely determined by the number of new loads entering the marketplace with lead time L. Constraint (3.6c) states that the number of bookings for any given lead time cannot exceed the number of loads that are currently available.

3.5.2 Convex Reformulation

Both the objective and constraint (3.6a) in the DTFM are nonconvex, and therefore we consider a convex reformulation below.

We make the following change of variables, similar to Section 3.4.2. Let

$$u_\ell \;=\; \frac{x_\ell\; v_\ell(p_\ell)}{\sum_{k=1}^{L} x_k\; v_k(p_k) + 1} \quad \text{for } \ell \geq 1, \quad \text{and} \quad u_0 \;=\; \frac{1}{\sum_{k=1}^{L} x_k\; v_k(p_k) + 1}. \tag{3.7}$$

Since $v_\ell(p_\ell) > 0$, $x_\ell = 0$ if and only if $u_\ell = 0$. Also,

$$\begin{aligned} \frac{u_\ell}{u_0} \;&=\; x_\ell\; v_\ell(p_\ell) \;=\; x_\ell\; \exp(\beta p_\ell + \beta_\ell^0) \\ \Rightarrow\; p_\ell \;&=\; \frac{1}{\beta}\left(\ln\left(\frac{u_\ell}{x_\ell u_0}\right) - \beta_\ell^0\right) \quad \text{for all } x_\ell > 0. \end{aligned} \tag{3.8}$$

Then the objective function in the DTFM becomes

$$\begin{aligned} &\mu \sum_{\ell=1}^{L} \frac{p_\ell\; x_\ell\; v_\ell(p_\ell)}{\sum_{k=1}^{L} x_k\; v_k(p_k) + 1} + C\; x_0 \\ \overset{(a)}{=}\;& \frac{\mu}{\beta}\left[\sum_{\ell=1}^{L} u_\ell \left(\ln\left(\frac{u_\ell}{x_\ell u_0}\right) - \beta_\ell^0\right)\right] + C\; x_0 \\ \overset{(b)}{=}\;& \frac{\mu}{\beta}\left[\sum_{\ell=1}^{L} u_\ell \left(\ln\left(\frac{u_\ell}{x_\ell}\right) - \beta_\ell^0\right) - (1-u_0)\ln(u_0)\right] + C\; x_0, \end{aligned}$$

where equality (a) follows from (3.7) and (3.8), and equality (b) holds because $\sum_{\ell=1}^{L} u_\ell + u_0 = 1$.

We define an extended real-valued function $g : \mathbb{R}_+^{L+1} \times [0,1]^{L+1} \mapsto \mathbb{R} \cup \{+\infty\}$ given by

$$g(\boldsymbol{x}, \boldsymbol{u}, u_0) \;:=\; \frac{\mu}{\beta}\left[\sum_{\ell=1}^{L} u_\ell \left(\ln\left(\frac{u_\ell}{x_\ell}\right) - \beta_\ell^0\right) - (1-u_0)\ln(u_0)\right] + C\; x_0.$$

We let $u_\ell \ln(u_\ell/x_\ell) = 0$ if $u_\ell = 0$, hence this term is continuous at any point where $u_\ell = 0$

and $x_\ell \geq 0$ (in particular, the continuity at $u_\ell = x_\ell = 0$ is implied by constraint (3.9d) that will be defined below). As a result, g is continuous on its domain. Moreover, it can be verified that g is convex.

Thus, the DTFM is reformulated as follows, which we refer to as the convex discrete-time fluid model (Convex DTFM):

$$\min_{(\boldsymbol{x},\boldsymbol{u},u_0)\in\mathbb{R}_+^{2(L+1)}} \left\{ \frac{\mu}{\beta} \sum_{\ell=1}^{L} u_\ell \left(\ln\left(\frac{u_\ell}{x_\ell}\right) - \beta_\ell^0 \right) - \frac{\mu}{\beta}(1-u_0)\ln(u_0) + C\, x_0 \right\} \tag{Convex DTFM}$$

$$\text{s.t.} \quad x_\ell - x_{\ell-1} = \mu u_\ell - \lambda \psi_{\ell-1} \qquad \forall \ell \geq 1 \tag{3.9a}$$

$$x_L = \lambda \psi_L \tag{3.9b}$$

$$\sum_{\ell=1}^{L} u_\ell + u_0 = 1 \tag{3.9c}$$

$$x_\ell \geq \mu u_\ell \qquad \forall \ell \geq 1. \tag{3.9d}$$

The Convex DTFM has a convex objective and linear constraints and can be solved efficiently. The DTFM and the Convex DTFM are equivalent, in the sense that given a feasible solution for one problem one can compute a feasible solution with the same objective value for the other problem using (3.7) and (3.8). Moreover, because the objective function is continuous and the feasible set is closed and bounded, an optimal solution exists by the extreme value theorem. Let $(\hat{\boldsymbol{x}}, \hat{\boldsymbol{u}}, \hat{u}_0)$ be an optimal solution for the Convex DTFM. Then an optimal price $\hat{\boldsymbol{p}}$ for the DTFM is given by

$$\hat{p}_\ell = \frac{1}{\beta}\left(\ln\left(\frac{\hat{u}_\ell}{\hat{x}_\ell \hat{u}_0}\right) - \beta_\ell^0 \right) \qquad \text{for } \ell \in \operatorname{supp}(\hat{\boldsymbol{x}}).$$

3.5.3 DTFM Gives a Lower Bound of Optimal Cost

We establish that the optimal objective value of the DTFM gives a lower bound for the long-run average cost per period for the MDP under any stationary policy. Without loss of generality, we restrict the state space of the MDP to the states communicating with the empty state $\boldsymbol{x} = \mathbf{0}$. (This set of states is the same for all policies — see Remark 3.1). This restricted state space is finite, since starting from the empty state, there can be at most L loads in the system at any time. Thus, under any stationary policy $\boldsymbol{\varphi}$, there exists a unique stationary distribution of the state $\boldsymbol{X}$. Then, for any stationary policy $\boldsymbol{\varphi}$, the long-run average cost per period is given by

$$R(\boldsymbol{\varphi}) \ := \ \mathsf{E}_{\boldsymbol{\varphi}}\left[\mu \sum_{\ell=1}^{L} \frac{\varphi_\ell(\boldsymbol{X})\ X_\ell\ v_\ell(\varphi_\ell(\boldsymbol{X}))}{\sum_{k=1}^{L} X_k\ v_k(\varphi_k(\boldsymbol{X}))\ +\ 1}\ +\ C\ X_0\right],$$

where the expectation is taken with respect to the stationary distribution of $\boldsymbol{X}$ under policy $\boldsymbol{\varphi}$.

Theorem 3.4. *The optimal objective value of the discrete-time fluid model* (DTFM) *is a lower bound for the long-run average cost per period of the* MDP *under any stationary policy* $\boldsymbol{\varphi}$*, i.e.,* $\hat{R} \leq R(\boldsymbol{\varphi})$ *for all* $\boldsymbol{\varphi}$*.*

The proof is given in Appendix B.2.

3.6 Open-Loop Pricing Policy and Its Asymptotic Optimality

In this section, we consider the pricing policy that applies in all states the prices obtained from an optimal solution of the discrete-time fluid model (DTFM). We show that this state-independent (open-loop) pricing policy achieves asymptotic optimality under a fluid scaling asymptotic regime. This provides theoretical support for using the open-loop pricing policy in practice. We also demonstrate the performance of the policy in numerical experiments.

3.6.1 Open-Loop Pricing Policy

We propose an open-loop pricing policy that sets a fixed price for each lead time, regardless of what other loads are available in the marketplace. The term "open-loop" indicates that the prices do not depend on the state of the system, as opposed to prices of a "closed-loop" policy. Note that, under an open-loop policy, the price of a given load may still change over time, because the lead time of a given load changes over time and the price of a load depends on the current lead time of the load.

The open-loop pricing policy considered uses the optimal prices for the DTFM. Specifically, let $(\hat{\boldsymbol{x}}, \hat{\boldsymbol{p}})$ be an optimal solution for the DTFM. The open-loop pricing policy $\hat{\boldsymbol{\varphi}}$ sets price $\hat{p}_\ell$ for all loads with lead time ℓ, regardless of the system state $\boldsymbol{x}$, i.e., $\hat{\varphi}_\ell(\boldsymbol{x}) = \hat{p}_\ell$ for all $\boldsymbol{x}$.

3.6.2 Asymptotic Regime

We evaluate policy $\hat{\boldsymbol{\varphi}}$ in the following asymptotic regime. Consider a sequence $\{\mathsf{MDP}^\theta\}$ of problem instances scaled by a factor $\theta = 1, 2, \cdots$. In instance MDP^θ, the arrival rates per unit clock time of loads and carriers are $\theta\lambda$ and $\theta\mu$, respectively. Therefore, the factor θ can be viewed as a measure of the market size. As the market size increases, the range of lead times, measured in clock time (as opposed to discrete time periods), remains unchanged. Without loss of generality, let this range be $[0, 1]$. The lead times of newly arriving loads are i.i.d. with cumulative distribution function $\Phi : [0, 1] \mapsto [0, 1]$.

In the instance MDP^θ, the lead time range $[0, 1]$ is partitioned into θ intervals, each with length $1/\theta$. In the discrete-time model MDP^θ, the length of "one period" — the length of one lead time interval — is equal to $1/\theta$ in clock time. Thus, the load arrival rate in each period (of length $1/\theta$) of the discrete-time model MDP^θ is λ, and the arrival rate per unit clock time is $\theta\lambda$. Given a new load arrival, the probability that the load has discrete lead time $\ell \in \{1, \ldots, \theta\}$ is given by $\psi_\ell^\theta := \Phi(\ell/\theta) - \Phi((\ell - 1)/\theta)$, and we let $\psi_0^\theta := 0$.

The multinomial logit model for the instance MDP^θ is as follows. The price sensitivity parameter is β for all θ. For the mean non-monetary utility, consider a continuous function $b_0 : [0, 1] \mapsto \mathbb{R}$. Let $b_{\max} := \max_{t \in [0,1]} b_0(t)$. For the instance MDP^θ, the parameter $\beta_{0,\ell}^\theta$ is given by $\beta^\theta_{0,\ell} := b_0(\ell/\theta)$. Thus the preference weights for the instance MDP^θ are given by

$$v^\theta_\ell(p) \;:=\; \exp(\beta p + \beta^\theta_{0,\ell}) \qquad \text{for } \ell = 1,\ldots,\theta.$$

In addition, as the market size scales up, the preference weight of the null alternative in the multinomial logit model also scales up. Otherwise, if the null alternative weight is a constant (e.g. 1), then the probability of choosing the null alternative will converge to zero as θ increases, and the problem will become trivial. Therefore, for the instance MDP^θ, the preference weight of the null alternative is θ. Hence, given state $\boldsymbol{x}^\theta := (x^\theta_\ell,\ \ell = 0,1,\ldots,\theta)$ and price $\boldsymbol{p}^\theta := (p^\theta_\ell,\ \ell = 1,\ldots,\theta)$, and given that a carrier checks the offers in the marketplace, the probability that the carrier books a load with lead time $\ell \in \{1,\ldots,\theta\}$ is given by

$$f^\theta_\ell(\boldsymbol{p}^\theta, \boldsymbol{x}^\theta) \;=\; \frac{x^\theta_\ell \; v^\theta_\ell(p^\theta_\ell)}{\sum_{k=1}^\theta x^\theta_k \; v^\theta_k(p^\theta_k) + \theta},$$

and the no-booking probability is given by

$$f^\theta_0(\boldsymbol{p}^\theta, \boldsymbol{x}^\theta) \;=\; \frac{\theta}{\sum_{k=1}^\theta x^\theta_k \; v^\theta_k(p^\theta_k) + \theta}.$$

3.6.3 Asymptotic Optimality

Consider any stationary pricing policy $\boldsymbol{\varphi}^\theta : \mathbb{N}^{\theta+1} \mapsto \mathbb{R}^\theta$ for MDP^θ. As shown in Section 3.5.3, there is a unique stationary distribution for the system state under policy $\boldsymbol{\varphi}^\theta$. Let $\boldsymbol{X}^\theta \in \mathbb{N}^{\theta+1}$ denote a random system state with distribution equal to the unique stationary distribution under policy $\boldsymbol{\varphi}^\theta$. Then, the long-run average cost per period of policy $\boldsymbol{\varphi}^\theta$ is given by

$$R^\theta(\boldsymbol{\varphi}^\theta) \;:=\; \mathsf{E}_{\boldsymbol{\varphi}^\theta}\left[\mu \sum_{\ell=1}^{\theta} \frac{\varphi^\theta_\ell(\boldsymbol{X}^\theta)\; X^\theta_\ell \; v^\theta_\ell(\varphi^\theta_\ell(\boldsymbol{X}^\theta))/\theta}{\sum_{k=1}^\theta X^\theta_k \; v^\theta_k(\varphi^\theta_k(\boldsymbol{X}^\theta))/\theta \;+\; 1} \;+\; C\; X^\theta_0\right]. \tag{3.10}$$

The open-loop pricing policy $\hat{\boldsymbol{\varphi}}^\theta$ for the instance MDP^θ is based on the following discrete-time fluid model:

$$\hat{R}^\theta \;:=\; \min_{\boldsymbol{x}^\theta \geq \boldsymbol{0},\, \boldsymbol{p}^\theta} \quad \mu \sum_{\ell=1}^{\theta} \frac{p_\ell^\theta \, x_\ell^\theta \, v_\ell^\theta(p_\ell^\theta)/\theta}{\sum_{k=1}^\theta x_k^\theta \, v_k^\theta(p_k^\theta)/\theta \;+\; 1} \;+\; C\, x_0^\theta \qquad (\theta\text{-scaled DTFM})$$

$$\text{s.t.} \quad x_\ell^\theta - x_{\ell-1}^\theta \;=\; \mu \frac{x_\ell^\theta v_\ell^\theta(p_\ell^\theta)/\theta}{\sum_{k=1}^\theta x_k^\theta v_k^\theta(p_k^\theta)/\theta \;+\; 1} \;-\; \lambda \psi_{\ell-1}^\theta \quad \forall \ell \geq 1 \quad (3.11a)$$

$$x_\theta^\theta \;=\; \lambda\, \psi_\theta^\theta \qquad (3.11b)$$

$$x_\ell^\theta \;\geq\; \mu \, \frac{x_\ell^\theta \, v_\ell^\theta(p_\ell^\theta)/\theta}{\sum_{k=1}^\theta x_k^\theta \, v_k^\theta(p_k^\theta)/\theta \;+\; 1} \quad \forall \ell \geq 1 \quad (3.11c)$$

Let $(\hat{\boldsymbol{x}}^\theta, \hat{\boldsymbol{p}}^\theta)$ denote an optimal solution to the θ-scaled DTFM. The open-loop policy $\hat{\boldsymbol{\varphi}}^\theta$ sets price $\hat{p}_\ell^\theta$ for all loads with lead time ℓ, i.e., $\hat{\varphi}_\ell^\theta(\boldsymbol{x}^\theta) = \hat{p}_\ell^\theta$ for all $\boldsymbol{x}^\theta$ and all ℓ. Note that

$$R^\theta(\hat{\boldsymbol{\varphi}}^\theta) \;\geq\; \inf_{\boldsymbol{\varphi}^\theta} R^\theta(\boldsymbol{\varphi}^\theta) \;\geq\; \hat{R}^\theta,$$

where the second inequality follows from Theorem 3.4. To establish the asymptotic optimality of policy $\hat{\boldsymbol{\varphi}}^\theta$, we bound the loss of policy $\hat{\boldsymbol{\varphi}}^\theta$ relative to the optimal objective value $\hat{R}^\theta$ of the θ-scaled DTFM, which is a lower bound for the optimal long-run average cost per period for MDP^θ, for all $\theta \in \mathbb{N}$. Policy $\hat{\boldsymbol{\varphi}}^\theta$ is *asymptotically optimal* if the *loss ratio* satisfies

$$\frac{R^\theta(\hat{\boldsymbol{\varphi}}^\theta) - \hat{R}^\theta}{\hat{R}^\theta} \;\to\; 0 \quad \text{as } \theta \;\to\; \infty. \qquad (3.12)$$

The following result shows that the prices $\hat{p}_\ell^\theta$ are decreasing in ℓ, and the bounds are used in the asymptotic optimality proof.

Lemma 3.5. *An optimal solution $(\hat{\boldsymbol{x}}^\theta, \hat{\boldsymbol{p}}^\theta)$ for the θ-scaled DTFM satisfies $\hat{x}_\ell^\theta \;\leq\; \lambda$ for $\ell = 0, \ldots, \theta$ and $\hat{p}_\ell^\theta \leq C$ for $\ell = 1, \ldots, \theta$. Also, preference weight $\hat{v}_\ell^\theta := v_\ell^\theta(\hat{p}_\ell^\theta) \leq K_v :=$ $\exp(\beta C + b_{\max})$ for all $\ell = 1, \ldots, \theta$ and all θ.*

The proof is given in Appendix B.3.

We make the following two assumptions to facilitate further analysis, one on the optimal prices for the θ-scaled DTFM and the other on the parameter μ.

Assumption 3.6. *The optimal prices for the θ-scaled DTFM are bounded away from 0. That is, there is a constant $\underline{p} > 0$ such that $\hat{p}_\ell^\theta \geq \underline{p}$ for all $\theta \in \mathbb{N}$ and all $\ell = 1, \ldots, \theta$.*

Assumption 3.7. *It holds that $\mu K_v < 1/4$.*

The first assumption requires that the output of the θ-scaled DTFM be sensible, as a solution with zero or negative price implies that model inputs are chosen inappropriately. The second assumption can be satisfied by choosing the length of each time period short enough, so that μ (i.e., the probability that a driver arrives in one period) is sufficiently small. We suspect that the condition of Assumption 3.7 is simply an artifact of our analysis technique, and the result in this section should hold in general without this assumption.

Theorem 3.8. *The open-loop pricing policy $\hat{\boldsymbol{\varphi}}^\theta$ is asymptotically optimal. More specifically, there is a constant $K > 0$ such that the loss ratio is bounded for all $\theta \geq 1$ by*

$$\frac{R^\theta(\hat{\boldsymbol{\varphi}}^\theta) - \hat{R}^\theta}{\hat{R}^\theta} \leq \frac{K}{\theta},$$

where K depends on the load and carrier arrival rates λ, μ, the choice model parameters β, $b_{\max}$, the penalty cost C, and the lower bound $\underline{p}$.

The result above confirms that the open-loop pricing policy $\hat{\boldsymbol{\varphi}}^\theta$ is effective when the market size is large. Although the asymptotic optimality of open-loop (static) pricing policies has been established for various problems in the revenue management literature, it is typical for the loss ratio, defined as in (3.12), to be $O(1/\sqrt{\theta})$ in these papers, e.g. [44, 52]. The intuition for these results follows from the central limit theorem: when the supply and demand processes are scaled up by θ, some $O(\sqrt{\theta})$ term due to the random variation in the system is ignored by the (deterministic) fluid approximation. Thus, it is surprising that open-loop policy in our model setting obtains a much better loss ratio of $O(1/\theta)$.

We present the key steps of the proof below to show the intuition of the $O(1/\theta)$ bound in the asymptotic optimality result. The proofs of supporting results are given in Appendix B.4.

3.6.4 Proof of Theorem 3.8

Consider a state variable $\boldsymbol{X}^\theta = (X_l^\theta, l = 0, 1, \ldots, \theta)$ of the θ-scaled system under the open-loop pricing policy $\hat{\boldsymbol{\varphi}}^\theta$. The first step of our proof is to show that under the stationary distribution, the variance of a weighted sum of the loads in the system is bounded by $O(\theta)$.

Lemma 3.9. *Suppose $\mu K_v < 1/4$. Let $\boldsymbol{X}^\theta$ denote the state vector under the stationary distribution of the open-loop pricing policy $\hat{\boldsymbol{\varphi}}^\theta$. For any $k = 1, \ldots, \theta$ and any sequence of constants $(a_\ell :\ 0 \leq a_\ell \leq 1,\ \ell = 1, \ldots, \theta)$, we have*

$$\mathsf{Var}\left(\sum_{\ell=k}^{\theta} a_\ell X_\ell^\theta\right) \leq (\theta - k + 1) K_c, \tag{3.13}$$

where $K_c := 2/(1 - 4\mu K_v)$.

We note that the proof of Lemma 3.9 would have been easy if X_j^θ and X_ℓ^θ are mutually independent for any $j \neq \ell$, as $\mathsf{Var}(X_\ell^\theta)$ can be bounded by an absolute constant (see Lemma B.3). However, the challenge is that because of drivers' choice behavior, X_j^θ and X_ℓ^θ are not mutually independent, and analyzing their covariance is very complicated. Our idea is to bound the variance of $\mathsf{Var}\left(\sum_{\ell=k}^{\theta} a_\ell X_\ell^\theta\right)$ directly, using an induction over the time index and the ergodicity of the Markov chain induced by the open-loop pricing policy $\hat{\boldsymbol{\varphi}}^\theta$.

Our next step is to bound the expected long-run average cost of the open-loop pricing policy. The one period cost function is given by

$$r^\theta(\boldsymbol{p}^\theta, \boldsymbol{x}^\theta) \ := \ \sum_{\ell=1}^{\theta} \mu \frac{p_\ell^\theta\ x_\ell^\theta\ v_\ell^\theta(p_\ell^\theta)/\theta}{\sum_{k=1}^{\theta} x_k^\theta\ v_k^\theta(p_k^\theta)/\theta\ +\ 1}\ +\ C\ x_0^\theta.$$

Recall that $(\hat{\boldsymbol{x}}^\theta, \hat{\boldsymbol{p}}^\theta)$ denotes an optimal solution to (θ-scaled DTFM). Let $\hat{v}_\ell^\theta := v_\ell^\theta(\hat{p}_\ell^\theta)$ for $\ell = 1, \ldots, \theta$. Then the long-run average cost per discrete period of instance MDP^θ

under policy $\hat{\boldsymbol{\varphi}}^\theta$ is given by

$$R^\theta(\hat{\boldsymbol{\varphi}}^\theta) \;=\; \mathsf{E}\left[r^\theta(\hat{\boldsymbol{p}}^\theta, \boldsymbol{X}^\theta)\right] \;=\; \mathsf{E}\left[\sum_{\ell=1}^{\theta} \mu \frac{\hat{p}_\ell^\theta\, X_\ell^\theta\, v_\ell^\theta(\hat{p}_\ell^\theta)/\theta}{\sum_{k=1}^{\theta} X_k^\theta\, v_k^\theta(\hat{p}_k^\theta)/\theta \;+\; 1} \;+\; C\, X_0^\theta\right]$$

$$= \mathsf{E}\left[\sum_{\ell=1}^{\theta} \mu \frac{\hat{p}_\ell^\theta\, X_\ell^\theta\, \hat{v}_\ell^\theta/\theta}{\sum_{k=1}^{\theta} X_k^\theta\, \hat{v}_k^\theta/\theta \;+\; 1} \;+\; C\, X_0^\theta\right],$$

where the expectations are with respect to the stationary distribution under the open-loop pricing policy $\hat{\boldsymbol{\varphi}}^\theta$.

Denote the mean of the state variable under the stationary distribution by $\bar{\boldsymbol{x}}^\theta = \mathsf{E}[\boldsymbol{X}^\theta]$. We would like to replace $\mathsf{E}\left[r^\theta(\hat{\boldsymbol{p}}^\theta, \boldsymbol{X}^\theta)\right]$ with $r^\theta(\hat{\boldsymbol{p}}^\theta, \bar{\boldsymbol{x}}^\theta)$, but they are not equal because the function $r^\theta(\hat{\boldsymbol{p}}^\theta, \boldsymbol{x}^\theta)$ is nonlinear. However, we can bound the difference between them using Lemma 3.9, by leveraging the structure of the MNL choice model and a trick from [72] that bounds the expected ratio of two random variables.

Lemma 3.10. *There is a constant $K_f := 2K_v^2 K_c > 0$, such that*

$$\left|\sum_{\ell=k}^{\theta} \mathsf{E}\left[f_\ell^\theta(\hat{\boldsymbol{p}}^\theta, \boldsymbol{X}^\theta)\right] - \sum_{\ell=k}^{\theta} f_\ell^\theta(\hat{\boldsymbol{p}}^\theta, \bar{\boldsymbol{x}}^\theta)\right| \;\le\; \frac{K_f}{\theta}, \quad \textit{for any } 1 \le k \le \theta,$$

and

$$\left|\mathsf{E}\left[r^\theta(\hat{\boldsymbol{p}}^\theta, \boldsymbol{X}^\theta)\right] - r^\theta(\hat{\boldsymbol{p}}^\theta, \bar{\boldsymbol{x}}^\theta)\right| \;\le\; \frac{\mu C K_f}{\theta}.$$

Using Lemma 3.10, we derive the last lemma we need, which bounds the difference between the mean system state under the open-loop pricing policy and the solution for the θ-scaled DTFM, i.e., $\bar{\boldsymbol{x}}^\theta$ and $\hat{\boldsymbol{x}}^\theta$. The proof of Lemma 3.11 again exploits the structure of θ-scaled DTFM, in particular its convex reformulation.

Lemma 3.11. *There is a constant $K_x := \mu(1+K_v)K_f > 0$ such that for all $\ell = 0, 1, \cdots, \theta$,*

$$|\bar{x}_\ell^\theta - \hat{x}_\ell^\theta| \;\le\; \frac{K_x}{\theta}.$$

Finally, putting everything together, we establish our main result in Theorem 3.8.

Proof of Theorem 3.8. Consider $R^\theta(\hat{\boldsymbol{\varphi}}^\theta)$ (the expected cost of the open-loop pricing policy) and $\hat{R}^\theta$ (the optimal objective value of the θ-scaled DTFM). We have

$$
\begin{aligned}
& R^\theta(\hat{\boldsymbol{\varphi}}^\theta) - \hat{R}^\theta \\
&= \mathsf{E}\left[r^\theta(\hat{\boldsymbol{p}}^\theta, \boldsymbol{X}^\theta)\right] \; - \; r^\theta(\hat{\boldsymbol{p}}^\theta, \hat{\boldsymbol{x}}^\theta) \\
&\leq \left|\mathsf{E}\left[r^\theta(\hat{\boldsymbol{p}}^\theta, \boldsymbol{X}^\theta)\right] - r^\theta(\hat{\boldsymbol{p}}^\theta, \bar{\boldsymbol{x}}^\theta)\right| \; + \; \left|r^\theta(\hat{\boldsymbol{p}}^\theta, \bar{\boldsymbol{x}}^\theta) - r^\theta(\hat{\boldsymbol{p}}^\theta, \hat{\boldsymbol{x}}^\theta)\right| \\
&\overset{(a)}{\leq} \frac{\mu C K_f}{\theta} \; + \; \left| C\,(\bar{x}_0^\theta - \hat{x}_0^\theta) \; + \; \mu \frac{\sum_{\ell=1}^\theta \hat{p}_\ell^\theta \, \bar{x}_\ell^\theta \, \hat{v}_\ell^\theta/\theta}{\sum_{k=1}^\theta \bar{x}_k^\theta \hat{v}_\ell^\theta/\theta \; + \; 1} \; - \; \mu \frac{\sum_{\ell=1}^\theta \hat{p}_\ell^\theta \, \hat{x}_\ell^\theta \, \hat{v}_\ell^\theta/\theta}{\sum_{k=1}^\theta \hat{x}_k^\theta \hat{v}_k^\theta/\theta \; + \; 1} \right| \\
&= \frac{\mu C K_f}{\theta} \; + \; \Bigg| C\,(\bar{x}_0^\theta - \hat{x}_0^\theta) \; + \; \frac{\mu}{\theta} \frac{\sum_{\ell=1}^\theta \hat{p}_\ell^\theta \, \hat{v}_\ell^\theta \, (\bar{x}_\ell^\theta - \hat{x}_\ell^\theta)}{\left(\sum_{k=1}^\theta \hat{v}_\ell^\theta \bar{x}_k^\theta/\theta \; + \; 1\right)\left(\sum_{k=1}^\theta \hat{x}_k^\theta \hat{v}_k^\theta/\theta \; + \; 1\right)} \\
&\qquad + \frac{\mu}{\theta^2} \frac{\left(\sum_{\ell=1}^\theta \hat{p}_\ell^\theta \, (\bar{x}_\ell^\theta - \hat{x}_\ell^\theta) \, \hat{v}_\ell^\theta\right)\left(\sum_{k=1}^\theta \hat{x}_k^\theta \hat{v}_k^\theta\right) - \left(\sum_{\ell=1}^\theta \hat{p}_\ell^\theta \, \hat{x}_\ell^\theta \, \hat{v}_\ell^\theta\right)\left(\sum_{k=1}^\theta (\bar{x}_k^\theta - \hat{x}_k^\theta)\hat{v}_k^\theta\right)}{\left(\sum_{k=1}^\theta \hat{v}_\ell^\theta \bar{x}_k^\theta/\theta \; + \; 1\right)\left(\sum_{k=1}^\theta \hat{x}_k^\theta \hat{v}_k^\theta/\theta \; + \; 1\right)} \Bigg| \\
&\leq \frac{\mu C K_f}{\theta} \; + \; C|\bar{x}_0^\theta - \hat{x}_0^\theta| \; + \; \frac{\mu}{\theta} \sum_{\ell=1}^\theta \hat{p}_\ell^\theta \, \hat{v}_\ell^\theta |\bar{x}_\ell^\theta - \hat{x}_\ell^\theta| \; + \; \frac{\mu}{\theta^2}\left(\sum_{\ell=1}^\theta \hat{p}_\ell^\theta \, \hat{v}_\ell^\theta \, |\bar{x}_\ell^\theta - \hat{x}_\ell^\theta|\right)\left|\sum_{k=1}^\theta \hat{x}_k^\theta \hat{v}_k^\theta\right| \\
&\qquad + \frac{\mu}{\theta^2}\left|\sum_{\ell=1}^\theta \hat{p}_\ell^\theta \, \hat{x}_\ell^\theta \, \hat{v}_\ell^\theta\right|\left(\sum_{k=1}^\theta \hat{v}_k^\theta |\bar{x}_k^\theta - \hat{x}_k^\theta|\right) \\
&\overset{(b)}{\leq} \frac{\mu C K_f}{\theta} \; + \; \frac{C K_x}{\theta} \; + \; \frac{\mu C K_v K_x}{\theta} \; + \; \frac{\mu\lambda(C K_v K_x)(\theta K_v)}{\theta^2} \; + \; \frac{\mu\lambda(\theta C K_v)(K_v K_x)}{\theta^2} \\
&= \frac{\mu C K_f}{\theta} \; + \; \frac{C K_x}{\theta} \; + \; \frac{\mu C K_v K_x}{\theta} \; + \; \frac{2\mu\lambda C K_v^2 K_x}{\theta},
\end{aligned}
$$

where step (a) uses Lemma 3.10 and the definition of function $r^\theta(\hat{\boldsymbol{p}}^\theta, \boldsymbol{x}^\theta)$, and step (b) uses Lemma 3.5 and Lemma 3.11.

Since all loads (arriving with rate λ per period) are either booked by some driver later or eventually expire, by Assumption 3.6 and Lemma 3.5, the optimal objective value of the θ-scaled DTFM is lower bounded by

$$\hat{R}^\theta \;\geq\; \lambda \min\left\{\underline{p}, C\right\} = \lambda \underline{p}.$$

Setting

$$K := \frac{\mu C K_f + C K_x + \mu C K_v K_x + 2\mu\lambda C K_v^2 K_x}{\lambda \underline{p}}$$

and putting everything together, we have

$$\frac{R^\theta(\hat{\boldsymbol{\varphi}}^\theta) - \hat{R}^\theta}{\hat{R}^\theta} \leq \frac{(\mu C K_f + C K_x + \mu C K_v K_x + 2\mu\lambda C K_v^2 K_x)/\theta}{\lambda \underline{p}} = \frac{K}{\theta}.$$

□

3.6.5 Numerical Experiments

Numerical experiment in this section will give further insight into the structure of the open-loop pricing policy and its asymptotic performance. We consider the following simulation setup. Let θ be the scaling factor in the associated asymptotic regime. The arrival probabilities of loads and drivers in each period are $\lambda = 0.2$ and $\mu = 0.3$ respectively. When a new load arrives, its initial lead time is θ. We consider a Homogeneous Preference setting under the multinomial logit model, with price coefficient $\beta = 0.5$ and non-monetary utility $\beta_0 = -5$. The penalty cost of an expiring load is $C = 20$.

Long-Run Average Cost:

We compare the long-run average cost between our proposed open-loop pricing policy (Static) and the optimal policy (Dynamic) obtained by solving the corresponding MDP. We conduct 100 replicates of simulation trials of 10,000 periods to compute the average per period cost under both policies while varying $\theta \in \{3, 9, 15\}$.

Figure 3.4 shows the simulated long-run average costs per period with 95% confidence intervals. We observe that the cost under the Static policy is very close to that under the Dynamic policy, with an opimality gap within 1.7% even for a small scaling factor ($\theta = 3$), and a an opimality gap of 0.6% for a moderate scaling factor ($\theta = 15$). We also report the theoretical lower bound of the average costs under any stationary policy derived from the θ-scaled DTFM, i.e., $\hat{R}^\theta$. It can be observed that the gap between the simulated costs under both policies and the lower bound is decreasing as we scale up the system.

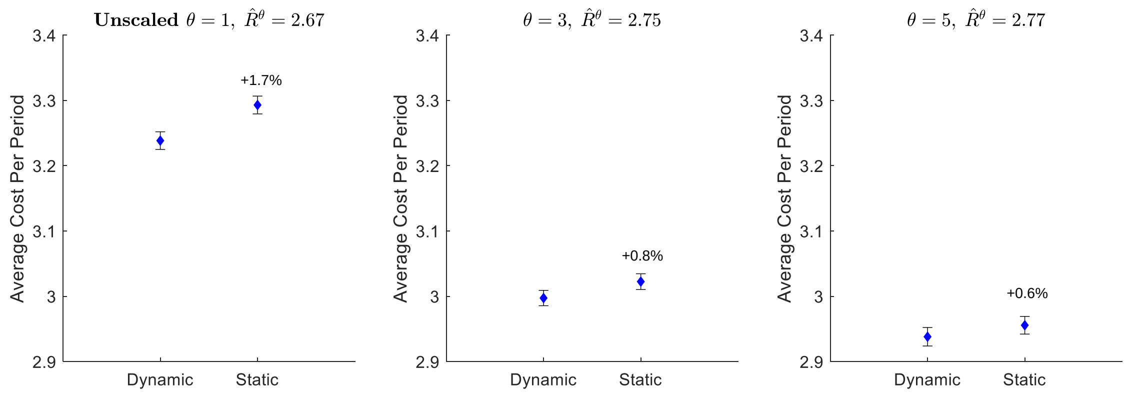

Figure 3.4: Long-run average costs under the (optimal) Dynamic and Static policies (95% confidence interval).

Order of the Loss Ratio:

We validate our conclusion in Theorem 3.8 that the loss ratio i.e., $\left(R^\theta(\hat{\boldsymbol{\varphi}}^\theta) - \hat{R}^\theta\right) \Big/ \hat{R}^\theta$, is of order $O(\theta^{-1})$. We conduct 100 replicates of simulation trials of $2,000 \times \theta$ periods, discarding the first half of the simulated periods and using the remainder to compute the average per period cost under the open-loop pricing policy $\hat{\boldsymbol{\varphi}}^\theta$ while varying $\theta \in \{10, 12, 15, 20, 30, 50, 100\}$. In Figure 3.5, the horizontal axis corresponds to $1/\theta$ and the vertical axis represents the loss ratio. We observe that our estimate on the order of the loss in Theorem 3.8 holds numerically for this setup. The solid line reports the fitted line of loss ratio against $1/\theta$, which is Loss Ratio $= 0.3451/\theta$, and the coefficient of determination $R^2 = 0.9769$.

Optimal Static Price Trajectory:

Figure 3.6 plots the optimal static prices as a function of the lead time for the θ-scaled DTFM, with scaling factor $\theta \in \{10, 20, 30\}$. The horizontal axis correspond to lead time and the vertical axis correspond to optimal static prices. The figure shows that the prices are strictly decreasing in lead time, which intuitively means that the platform has to offer a higher price if a load gets closer to the expiration time. Moreover, as the scaling factor θ increases, we observe that the price curves from the θ-scaled DTFM converge to a smooth

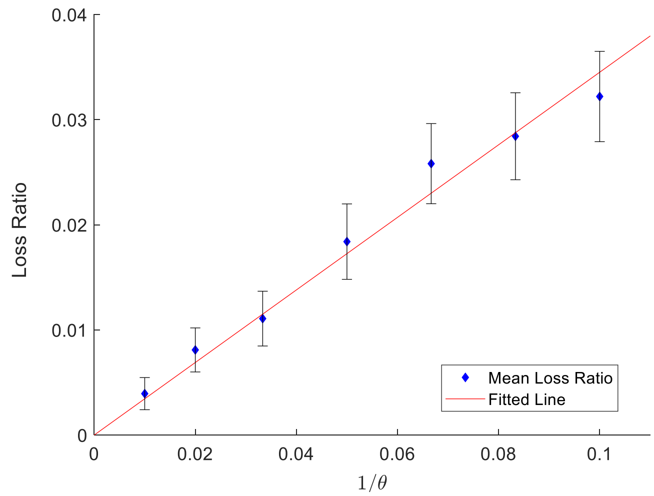

Figure 3.5: The order of loss ratio as a function of the scaling factor θ (95% confidence interval).

price trajectory. This motivates us to study a continuous-time fluid model (CTFM), which is presented in the next section.

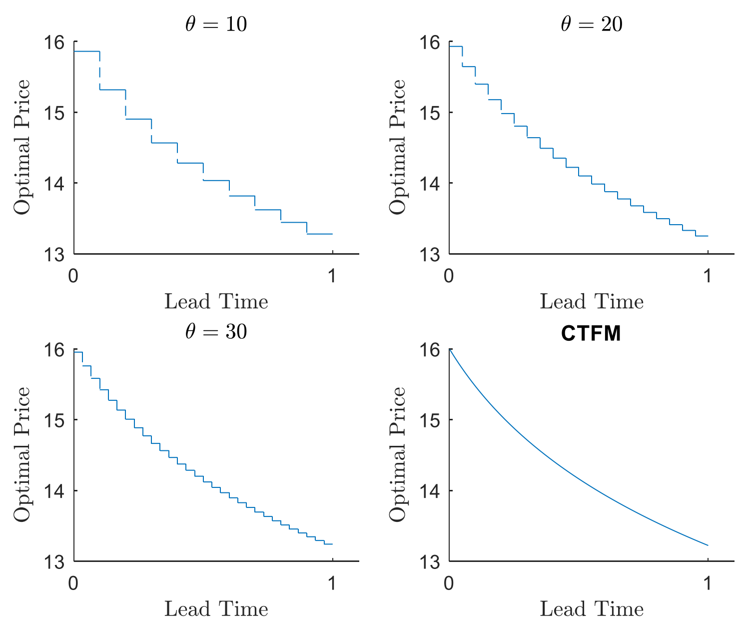

Figure 3.6: Optimal static prices for different scaling factors (θ).

3.7 Continuous-Time Fluid Model

While the convex discrete-time fluid model (Convex DTFM) is tractable, it is not easy to analyze the final solution explicitly as a function of the lead time. Inspired by the θ-scaled discrete-time fluid model (θ-scaled DTFM) for the θ-scaled stochastic system as discussed in Section 3.6, we introduce a continuous-time fluid model (CTFM) in this section. We will show that the CTFM can be transformed into an optimal control problem with linear constraints and a convex objective. Analyzing the resulted optimal control problem provides managerial insights in the pricing problem and the associated open-loop pricing policy.

3.7.1 Formulation

As discussed in Section 3.6.2, we normalize the range of possible lead time to $[0, 1]$. Recall that function Φ represents the CDF of the initial lead time of a load. We further assume Φ is a continuous distribution, possibly with a point mass on 1. Let $\Phi(1-) = \lim_{x\to 1^-} \Phi(x)$. We let $\phi : [0, 1) \mapsto \mathbb{R}_+$ be the corresponding probability density function, which is assumed to be continuous. Also, recall that $b_0 \in \mathcal{C}^0$ represents the non-monetary utility for a load with respect to its lead time. We further define a continuous preference weight function $v : [0, 1] \times \mathbb{R} \mapsto \mathbb{R}_+$ given by

$$v(\tau, p) \;=\; \exp(\beta p + b_0(\tau)),$$

where τ represents the continuous lead time, p is the offered price, and β is the price sensitivity coefficient. Similar to the Homogeneous Preference setting with discrete lead time, if drivers are indifferent to lead time, we have $b_0(\tau) \equiv \beta_0$ for some β_0. When the context is clear, we also refer to such setting with continuous lead time as the Homogeneous Preference setting.

Revisiting the θ-scaled DTFM and letting the length of each lead time period $\Delta\tau^\theta = 1/\theta \to 0$ as $\theta \to \infty$, we derive a deterministic optimization problem with continuous lead time. More specifically, we let $x : [0,1] \mapsto \mathbb{R}_+$ be the deterministic system state, which represents a load intensity function over the interested range of continuous lead time. One can imagine a system of fluid flowing from location $\tau = 1$ to location $\tau = 0$, where $x(\tau)$ corresponds to the volume of fluid that flows by location τ instantaneously. We let $p : [0,1] \mapsto \mathbb{R}$ be the system control, which represents the offered price over lead time. We further confine state x to be continuously differentiable, i.e., $x \in \mathcal{C}^1$, and control p to be continuous. Letting $\dot{x}$ represent the first order derivative of function x, we formulate the continuous-time fluid model (CTFM) as follows:

$$\min_{x \in \mathcal{C}^1,\, p \in \mathcal{C}^0} \quad \int_0^1 \mu \frac{p(\tau)\, x(\tau)\, v(\tau, p(\tau))}{\int_0^1 x(\tau')\, v(\tau', p(\tau'))\, \mathrm{d}\tau' + 1}\, \mathrm{d}\tau \;+\; C\, x(0) \tag{CTFM}$$

$$\text{s.t.} \quad \dot{x}(\tau) \;=\; \mu \frac{x(\tau)\, v(\tau, p(\tau))}{\int_0^1 x(\tau')\, v(\tau', p(\tau'))\, \mathrm{d}\tau' + 1} - \lambda\, \phi(\tau) \qquad \forall \tau \in [0,1], \tag{3.14a}$$

$$x(1) \;=\; \lambda\, (1 - \Phi(1-)) \tag{3.14b}$$

$$x(\tau) \;\geq\; 0 \qquad \forall \tau \in [0,1]. \tag{3.14c}$$

The objective of the model is to minimize the instantaneous cost of the underlying deterministic system. The integral term corresponds to the expected instantaneous payment to drivers as they choose loads under the multinomial logit model, and the term $C\; x(0)$ represents the instantaneous penalty cost incurred by expiring loads. Constraint (3.14a) is the state equation, which governs the dynamics of the system state, representing the rate of change in load intensity over lead time. There are two opposing forces that decide the rate of change in load intensity, i.e., drivers booking available loads and new loads arriving into the system. This constraint takes the form of an ordinary differential equation. We note that its right hand side involves an integral term over the range of lead time, which makes the constraint complex. Constraint (3.14b) defines the boundary condition on the system state. Constraint (3.14c) is the nonnegativity constraint on the state.

3.7.2 Alternative Formulation as An Optimal Control Problem

The CTFM seems difficult to solve at first glance. So, we change its variables to transform the CTFM into a more tractable formulation. We follow a similar routine as described in Section 3.5.2. We omit the details of the deduction process but report below the final formulation.

We let $x : [0,1] \mapsto \mathbb{R}_+$ be the system state, and let $u : [0,1] \mapsto \mathbb{R}_+$ and $u_0 \in \mathbb{R}_+$ be the control. We further confine $x \in \mathcal{C}^1$ and $u \in \mathcal{C}^0$. Then, an alternative formulation to the CTFM is given by

$$\min_{\substack{x\in\mathcal{C}^1,\, u\in\mathcal{C}^0,\\ u_0\in\mathbb{R}_+}} \quad \frac{\mu}{\beta}\int_0^1 u(\tau)\left(\ln\left(\frac{u(\tau)}{x(\tau)}\right) - b_0(\tau)\right)\mathrm{d}\tau - \frac{\mu}{\beta}(1-u_0)\ln(u_0) + C\,x(0) \tag{3.15a}$$

$$\text{s.t.} \quad \dot{x}(\tau) \;=\; \mu\, u(\tau) - \lambda\,\phi(\tau) \qquad \forall \tau \in [0,1], \tag{3.15b}$$

$$x(1) \;=\; \lambda\,(1-\Phi(1-)) \tag{3.15c}$$

$$x(\tau) \;\geq\; 0, \quad u(\tau) \;\geq\; 0 \qquad \forall \tau \in [0,1], \tag{3.15d}$$

$$\int_0^1 u(\tau)\mathrm{d}\tau + u_0 \;=\; 1. \tag{3.15e}$$

Although problem (3.15) is convex, it is an optimal control problem with complex path constraint (3.15e). So, we further introduce an auxiliary state variable $y : [0,1] \mapsto \mathbb{R}_+$ such that $y \in \mathcal{C}^1$, and translate constraint (3.15e) into

$$\dot{y}(\tau) \;=\; -\,u(\tau) \quad \text{with} \quad y(0) \;=\; 1.$$

Then problem (3.15) is reformulated as

$$\min_{\substack{x\in\mathcal{C}^1,\, y\in\mathcal{C}^1\\ u\in\mathcal{C}^0}} \quad \frac{\mu}{\beta}\int_0^1 u(\tau)\left(\ln\left(\frac{u(\tau)}{x(\tau)}\right) - b_0(\tau)\right)\mathrm{d}\tau - \frac{\mu}{\beta}(1-y(1))\ln(y(1)) + C\,x(0) \tag{3.16a}$$

$$
\begin{aligned}
\text{s.t.} \quad \dot{x}(\tau) \;&=\; \mu\, u(\tau) - \lambda\, \phi(\tau) && \forall \tau \in [0,1], && (3.16b)\\
\dot{y}(\tau) \;&=\; -\, u(\tau) && \forall \tau \in [0,1], && (3.16c)\\
x(1) \;&=\; \lambda\,(1 - \Phi(1-)), \quad y(0) \;=\; 1 && && (3.16d)\\
x(\tau) \;&\geq\; 0, \quad y(\tau) \;\geq\; 0, \quad u(\tau) \;\geq\; 0 && \forall \tau \in [0,1], && (3.16e)
\end{aligned}
$$

where constraints (3.16b) and (3.16c) are state equations, describing the system dynamics together with boundary condition (3.16d); constraint (3.16e) ensures nonnegativity on the state and control variables. We note that problem (3.16) is in the form of an optimal control problem with linear constraints and a convex objective.

Before further analyzing the structure of the optimal solution for problem 3.16, we discuss how to recover the optimal price for the (CTFM) and the corresponding open-loop pricing policy. Letting $(x^*,\ y^*,\ u^*)$ be the optimal solution for the CTFM and $u_0^* = y^*(1)$, we can recover the optimal price for the CTFM by computing

$$
p^*(\tau) \;=\; \frac{1}{\beta}\left(\ln\left(\frac{u^*(\tau)}{x^*(\tau)\, u_0^*}\right) - b_0(\tau)\right). \tag{3.17}
$$

Similar to the discussion in Section 3.6, we can use p^* to set up a open-loop pricing policy that offers a fixed price for loads with any continuous lead time. The open-loop pricing policy can be regarded as a limiting case of the pricing policy for the θ-scaled stochastic system as we let $\theta \to \infty$, which achieves near optimal performance.

3.7.3 Structure of the Optimal Solution

To analyze the structural of the optimal solution for problem (3.16), we let $\pi_1 : [0,1] \mapsto \mathbb{R}$ and $\pi_2 : [0,1] \mapsto \mathbb{R}$ be the *costate* variables to the state equations (3.16b) and (3.16c). The costate variables can be viewed as Lagrange multipliers associated with the state equations or the shadow prices [73, 74]. A solution is optimal for the optimal control problem (3.16) if its Hamiltonian and the costate variables satisfy certain conditions, see details in Appendix B.5 and [75].

We first discuss the properties of the optimal solution under a general initial lead time CDF Φ. While a closed-form expression for the optimal solution is difficult to find, these properties provide insights into the structure of the optimal solution and are interesting in themselves.

Theorem 3.12. *Suppose an optimal solution* $(x^*,\ y^*,\ u^*)$ *for problem* (3.16) *exists. Let* $(x^*,\ p^*)$ *be the corresponding optimal solution for the* CTFM*. We have the following statements:*

a) The solution $(x^*,\ y^*,\ u^*)$ *is a global optimal solution.*

b) There is a costate π_1^* *to the state equation* (3.16b) *such that* π_1^* *is increasing with* $\pi_1^*(0) = -C$.

c) There is a costate π_2^* *to the state equation* (3.16c) *such that* $\pi_2^* \equiv \frac{1-u_0^*}{u_0^*} - \ln(u_0^*)$ *for some* $u_0^* \in [0, 1]$.

d) The optimal price trajectory p^* *is decreasing.*

By Theorem 3.12, we know that the optimal price trajectory is decreasing in lead time. Recall that in Theorem 3.3, we require Homogeneous Preference for the optimal price for the MDP to be decreasing in lead time *in each state*, and the property does not generally hold when we consider Heterogeneous Preference. Here, in the fluid model, we are optimizing a static price trajectory from an averaging perspective. So, regardless of drivers' preference toward lead time, the platform should always post a higher price for loads with a shorter lead time under the open-loop pricing policy, which is consistent with our intuition.

Next, we discuss an interesting special case, in which all arriving loads have the same lead time and drivers have homogeneous preference toward lead time. A *closed-form expression* for the optimal solution can be obtained for this case. Surprisingly, in the optimal solution, we can show that the loads remaining in the system will decrease at a *linear* rate once they arrive to the platform.

Corollary 3.13 (Same Initial Lead Time)**.** *In the Homogeneous Preference setting, i.e., $b_0(\tau) \equiv \beta_0$, and with all new loads arriving with lead time equal to 1, i.e., $\Phi(1-) = 0$, an optimal solution $(x^*,\ y^*,\ u^*)$, if existing, for problem (3.16) is given in the following form:*

$$x^*(\tau) = [\lambda - \mu(1-u_0^*)] + \mu(1-u_0^*)\tau, \quad y^*(\tau) = 1-(1-u_0^*)\tau, \quad \textit{and} \quad u^*(\tau) = 1-u_0^*,$$

with the costate variables

$$\pi_1^*(\tau) = \frac{1}{\beta}\ln\left(\frac{[\lambda - \mu(1-u_0^*)] + \mu(1-u_0^*)\tau}{\lambda - \mu(1-u_0^*)}\right) - C \ \textit{and}\ \pi_2^* \equiv \frac{\mu}{\beta}\left[\ln(u_0^*) - \frac{1-u_0^*}{u_0^*}\right],$$

where $u_0^* \in (0,1)$ and $\lambda > \mu(1-u_0^*)$. And, u_0^* is determined by solving a convex program.

3.7.4 Numerical Experiment

We use simulation to show how different arrival and preference patterns change the solutions for the CTFM. Throughout the section, we assume the arrival rates of loads and drivers are $\lambda = 0.2$ and $\mu = 0.3$ respectively. The penalty cost of an expiring load is $C = 20$.

Different Arrival Patterns:

We consider three different settings on the initial lead time of a new load. *Setting A:* All loads arrive with the largest lead time, i.e., $\Phi(1-) = 0$ and $\Phi(1) = 1$. *Setting B:* The initial lead time is uniformly distributed, i.e., $\Phi(\tau) = \tau$ for $\tau \in [0,1]$. *Setting C:* The initial lead time is distributed with cumulative distribution function $\Phi(\tau) = \tau^2$ for $\tau \in [0,1]$. Drivers are indifferent toward lead time with preference weight $v(\tau,p) = \exp(0.5 \times p - 5)$. Figure 3.7 shows the optimal state and price trajectories for the CTFM in the above settings.

As shown by Corollary 3.13, the optimal state trajectory in Setting A is a straight line. Interestingly, in Settings B and C, we observe that the optimal state trajectories first increase than decrease, which reflects the balance between new loads arriving and drivers booking loads over the lead time. In all three settings, the optimal prices are decreasing with lead time, which validates property d) of Theorem 3.12.

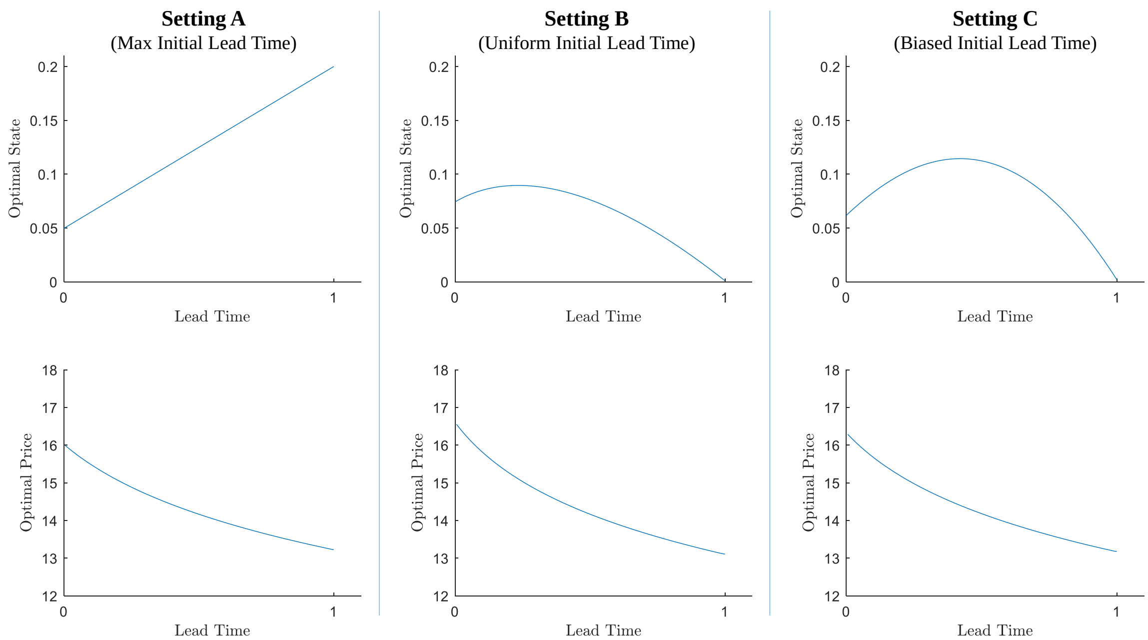

Figure 3.7: CTFM solutions with different load arrival patterns

__**Different Preference Patterns**__:

The initial lead time of a new load follows a triangle distribution with mode 0.5 over the range $[0, 1]$. We consider three different settings on the preference weight function $v(\tau, p)$ over lead time $\tau \in [0, 1]$. *Setting I:* Drivers prefer loads with a shorter lead time, with preference weight $v(\tau, p) = \exp(0.5 \times p - 4 - 2\tau)$. *Setting II:* Drivers are indifferent toward lead time with preference weight $v(\tau, p) = \exp(0.5 \times p - 5)$, i.e., the Homogeneous Preference setting. *Setting III:* Drivers prefer loads with a longer lead time, with preference weight $v(\tau, p) = \exp(0.5 \times p - 6 + 2\tau)$. We note that the average values of the non-monetary utility over lead time in the above three settings are the same.

Figure 3.8 shows the optimal state and price trajectories for the CTFM in the above settings. We observe that as drivers prefer loads with a shorter lead time, the state trajectory is biasing leftward with a higher peak, and the price trajectory is biasing downward, which implies a lower average cost.

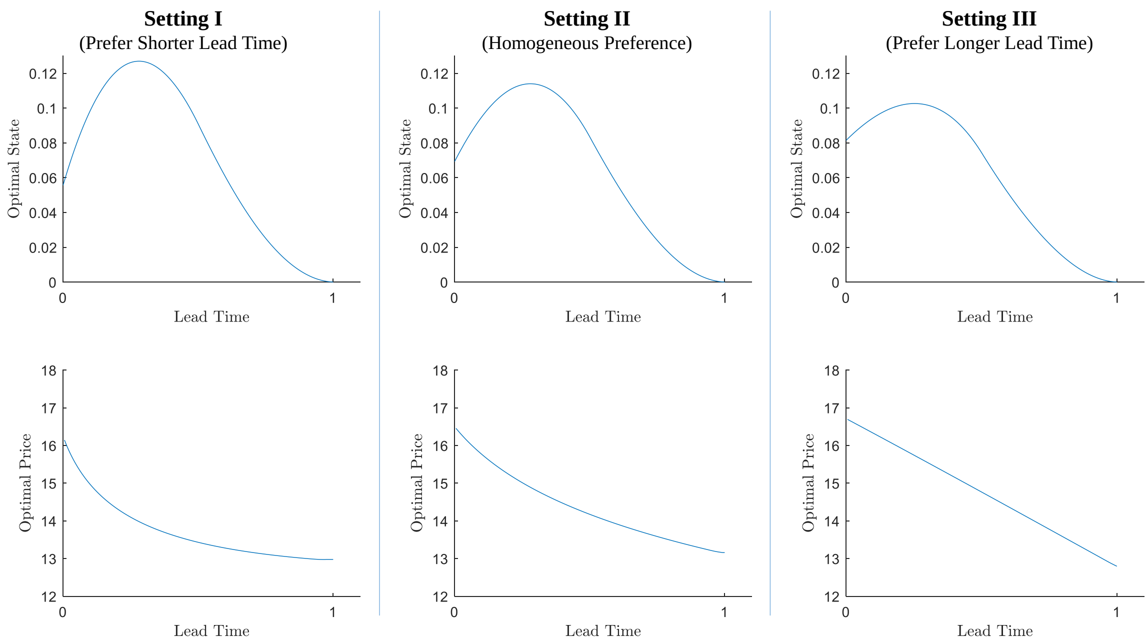

Figure 3.8: CTFM solutions with different driver preference patterns

3.8 Extensions

As shown in Section 3.7, the continuous-time fluid model (CTFM) provides a unified formulation to set a open-loop pricing policy if we consider continuous lead time. Established tools in the optimal control theory can be used to analyze and solve this fluid model. In this section, we further consider some practical settings arisen in the implementations for pricing in the truckload marketplace. We show that the CTFM can be easily extended to accommodate these settings. We note that all the resulted formulations can be further transformed into tractable formulations using similar tricks as discussed in Section 3.7.2, so we only present below the formulations with the original control variable on price.

3.8.1 Heterogeneous Load Types

In the previous sections, for simplicity, we considered pricing for loads with a homogeneous type (e.g., same origin and destination). However, both the MDP and the fluid models can easily incorporate loads of different types (e.g., different origins and destinations), if we assume all loads can be categorized into finitely many types. Here we consider the CTFM studied in Section 3.7.

Let $q = 1, \ldots, Q$ index the (finite) load types. Correspondingly, we add one dimension on load type, represented by subscript q, to the state variable x, the control variable p, the preference weight v, penalty cost C and parameters λ, Φ, ϕ. This leads to the following fluid model with Q load types:

$$
\begin{aligned}
\min_{\substack{x_q \in \mathcal{C}^1,\, p_q \in \mathcal{C}^0, \\ q=1,\ldots,Q}} \quad & \mu \, \frac{\int_0^1 \sum_{q=1}^{Q} p_q(\tau) x_q(\tau) v_q(\tau, p_q(\tau)) \mathrm{d}\tau}{\int_0^1 \sum_{q=1}^{Q} x_q(\tau') v_q(\tau', p_q(\tau')) \mathrm{d}\tau' + 1} \; + \; \sum_{q=1}^{Q} C_q \, x_q(0) \\
\text{s.t.} \quad & \dot{x}_q(\tau) \; = \; \mu \, \frac{x_q(\tau) v_q(\tau, p_q(\tau))}{\int_0^1 \sum_{q=1}^{Q} x_q(\tau') v_q(\tau', p_q(\tau')) \mathrm{d}\tau' + 1} - \lambda_q \, \phi_q(\tau) \\
& \hspace{16em} \text{for } \tau \in [0,1], \; q = 1, \ldots, Q, \\
& x_q(1) \; = \; \lambda_q \, (1 - \Phi_q(1-)) \hspace{8em} \text{for } q = 1, \ldots, Q, \\
& x_q(\tau) \; \geq \; 0 \hspace{10em} \text{for } \tau \in [0,1], \; q = 1, \ldots, Q.
\end{aligned}
$$

3.8.2 Periodic Arrival Patterns

For the the basic model, we assume that the load demand rate λ and the driver supply rate μ are constants in the long-run. In practice, the truckload market faces strong seasonality patterns in both load shipment demand and trucker supply [76]. Let T_λ and T_μ denote the periods of the loads and the drivers arrival patterns respectively and assume $T_\lambda = T_\mu = T$ without loss of generality. We note that we can let T be sufficiently large so the setting handles general arrival patterns.

Adding a new dimension of clock time, indexed by t, to the corresponding variables and parameters in the CTFM, we have the following formulation for periodic arrival patterns:

$$
\min_{x \in \mathcal{C}^1,\, p \in \mathcal{C}^0} \quad \int_0^T \mu(t) \, \frac{\int_0^1 p(\tau,t) x(\tau,t) v(\tau, p(\tau,t)) \mathrm{d}\tau}{\int_0^1 x(\tau',t) v(\tau', p(\tau',t)) \mathrm{d}\tau' + 1} \, \mathrm{d}t \; + \; C \int_0^T x(0,t) \, \mathrm{d}t \qquad (3.18a)
$$

$$\text{s.t.} \quad \frac{\partial\, x(\tau,t)}{\partial\,(\tau,-t)} \;=\; \mu(t)\, \frac{x(\tau,t)v(\tau,p(\tau,t))}{\int_0^1 x(\tau',t)v(\tau',p(\tau',t))\mathrm{d}\tau' + 1} \;-\; \lambda(t)\,\phi(\tau)$$

$$\text{for } \tau \in [0,1],\ t \in [0,T], \tag{3.18b}$$

$$x(1,t) \;=\; \lambda(t)\,(1 - \Phi(1-)) \qquad \text{for } t \in [0,T], \tag{3.18c}$$

$$x(\tau,0) \;=\; x(\tau,T), \quad p(\tau,0) \;=\; p(\tau,T) \qquad \text{for } \tau \in [0,1], \tag{3.18d}$$

$$x(\tau,t) \;\geq\; 0 \qquad \text{for } \tau \in [0,1],\ t \in [0,T]. \tag{3.18e}$$

In the above formulation, the objective (3.18a) minimizes the total cost incurred within a cycle of T units of clock time. Constraint (3.18b) represents the dynamics of the system, which is defined by a directional derivative over state x. The direction taken by the directional derivative has opposing signs before lead time τ and clock time t, since the lead time of any load decreases as time passes. Constraint (3.18c) sets the boundary condition on state x. Constraint (3.18d) ensures that the values of the state x at the beginning and the end of the cycle match, and so do the values of the control p. Constraint (3.18e) enforces nonnegativity.

3.9 Concluding Remarks

We study a dynamic pricing problem for online truckload marketplace where carriers choose among available loads on the platforms. We formulate the problem as an infinite horizon, discrete-time, average cost Markov decision process and incorporate a multinomial logit model to depict carriers' choice behavior. We consider a deterministic approximation of the problem and formulate a discrete-time fluid model (DTFM). A near optimal pricing policy based on the solution of the DTFM is proposed, which exhibits a loss ratio of order $O(1/\theta)$ when the market size is scaled by θ. We further formulate a continuous-time fluid model (CTFM) and discuss several extensions.

There are several potential avenues for future research. In our study, the arrivals of new loads are exogenous, i.e., how we charge shippers does not affect the load arrivals. One can relax this assumption and study the corresponding pricing problems. Also, we assume that drivers are myopic and do not wait in the system for prices to change. It would be interesting to examine settings where drivers are strategic to the pricing policy and find remedies to combat strategic behaviors. Lastly, we put less emphasis on the calibration of the underlying choice model. It would be helpful to work with the industry to validate the eligibility of the multinomial logit model or find alternative choice models.

CHAPTER 4
REVENUME MANAGEMENT UNDER A MIXTURE OF MULTINOMIAL LOGIT AND INDEPENDENT DEMAND MODELS

4.1 Overview

A growing body of literature indicates that using choice models to capture customer substitution between products can provide significant improvements in the revenues [39, 77]. However, an inherent tension is involved in picking a choice model with which to capture the choice process of the customers. A more sophisticated choice model may capture the choice process of the customers more faithfully, whereas a simpler choice model may result in tractable optimization problems when finding the optimal assortment of products to offer or prices to charge.

We consider assortment optimization problems under a mixture of multinomial logit and independent demand models. The multinomial logit model is arguably one of the most prevalent choice models for capturing customer choice behavior. In the multinomial logit model, each customer associates a random utility with each product and the no-purchase option, choosing the available alternative with the largest utility. The independent demand model has been a reliable workhorse, which is relatively simple to estimate and often yields tractable models for making operational decisions [78]. In the independent demand model, a customer arrives into the system with a particular product in mind. If this product is unavailable, then she leaves without a purchase. In this chapter, we mix these two very common demand models, which is, perhaps, the most natural approach to combine the representational power of both models. Some customers make a purchase under the multinomial logit model, whereas others do so under the independent demand model. The demand emerges as a mixture of these two customer segments.

Main Contributions: We give algorithms for numerous assortment problems, characterize the structure of optimal assortments, and check the prediction effectiveness of our choice model.

Assortment Optimization. In the single-shot assortment optimization problem, we have a certain revenue for each product. Customers choose among the offered products according to our mixture choice model. The goal is to find an assortment of products that maximizes the expected revenue obtained from a customer. We show that we can solve a polynomial-sized linear program (LP) to find the optimal assortment (Theorem 4.2). Thus, the assortment optimization problem under our mixture choice model is efficiently solvable. Assortment optimization problems under mixtures of choice models are notoriously difficult. For example, the assortment optimization problem under a mixture of just two multinomial logit models is *NP-hard* [29]. To our knowledge, we are the first to give an efficient method for assortment optimization under a mixture of choice models.

Combinatorial Algorithm. We show that if a product, all else being equal, has a larger purchase probability in the independent demand model or a smaller preference weight in the multinomial logit model, then it becomes more attractive to offer in the optimal assortment (Theorem 4.3). Besides shedding light on the structure of the optimal assortment, this result allows us to give a combinatorial algorithm for assortment optimization. Although a combinatorial algorithm exists, our LP formulation ultimately becomes useful for network revenue management.

Network Revenue Management. We consider assortment-based network revenue management problems, where we have resources with limited capacities and the sale of each product consumes a combination of resources. The goal is to find a policy for deciding which assortment of products to offer to each arriving customer to maximize the total expected revenue over a finite selling horizon. We consider a previously proposed LP approximation in which the decision variables are the probabilities with which we offer each subset of products to the customers. Thus, the number of decision variables increases exponentially with the number of products. We show that if the customers choose according to our mixture choice model, then we can immediately reduce the LP approximation to a compact LP whose numbers of decision variables and constraints increase only quadratically with the number of products (Theorem 4.4). We show that we can recover an optimal solution to the original LP approximation by using an optimal solution to the compact LP (Theorem 4.6). Lastly, in our computational experiments, we demonstrate that our compact LP substantially reduces the computation times.

4.2 Related Literature

The optimal assortment under the multinomial logit model is nested by revenue [16, 15], including a certain number of products with the largest revenues. This structure does not hold under our mixture choice model. The assortment optimization problem under a mixture of multinomial logit models is NP-hard even when there are only two multinomial logit models in the mixture [28, 79, 29]. The authors give approximation schemes and integer programming formulations. Also, it is NP-hard to approximate the problem within a factor of $O(1/m^{1-\epsilon})$ for any $\epsilon > 0$ [80], where m is the number of multinomial logit models in the mixture.

Researchers have developed LP formulations for assortment optimization problems, e.g., [18] for the generalized attraction model and [30] for the Markov chain choice model. One can build on these LP formulations to obtain compact LP formulations for network revenue management problems. The multinomial logit model is a special case of both the generalized attraction and Markov chain choice models, but our mixture of multinomial logit and independent demand models is not a special case of these choice models. Thus, we resort to entirely different techniques to obtain the LP formulations in this chapter. A compact formulation is given in [22] for a *nonlinear* program that appears when jointly making product stocking and assortment decisions under the multinomial logit model. In [81], the authors give an LP formulation for assortment optimization under the multinomial logit model when there are constraints on the offered assortment that can be captured by a totally unimodular constraint matrix.

Motivated by online retail, in which customers examine search results page by page, [82], [83] and [84] develop extensions of the multinomial logit model that allow the customers to incrementally view the products in batches. The authors give algorithms for finding the optimal sequence of product batches to offer. Consideration sets are incorporated in [85], [86] and [87], where each customer focuses only on the set of products in her consideration set and chooses within the consideration set under the multinomial logit model. Dynamic assortment optimization problems under the multinomial logit model are considered in [88] and [89], where the assortments offered to the customers are dictated by the inventory remaining on the shelf. We focus our literature review on the multinomial logit model, but assortment optimization has been studied under other choice models. For representative approaches, we refer to [31], [90], [91], and [92] for rank-based choice model, [93] for the Markov chain choice model, [25], [94], [27], and [95] for the nested logit model, and [96] for the paired combinatorial logit model.

Incorporating customer choice into network revenue management problems is an active area of research. LP approximations are given in [16] and [17] for these problems. The number of decision variables in their LP approximation increases exponentially with the number of products. Under our mixture choice model, we are able to reduce the size of their LP dramatically. Other approaches to these problems are based on approximating the value functions. For such approaches, we refer to reference [97, 19, 98, 99, 100].

4.3 Problem Formulation

The set of products is $N = \{1, \ldots, n\}$. There are two customer segments. The customers in the first segment make a purchase according to the multinomial logit model. In the multinomial logit model, we use $v_i > 0$ to denote the preference weight of product i. We normalize the preference weight of the no-purchase option to one. We let $V(S) = \sum_{i \in S} v_i$

to capture the total preference weight of the products in the subset $S \subseteq N$. In this case, if we offer the subset $S \subseteq N$ of products, then a customer in the first segment purchases product $i \in S$ with probability $v_i/(1+V(S))$. The customers in the second segment make a purchase according to the independent demand model. In the independent demand model, we use $\theta_i > 0$ to denote the probability that a customer is interested in product i. In this case, if we offer the subset $S \subseteq N$ of products, then a customer in the second segment purchases product $i \in S$ with probability θ_i. The probability that an arriving customer is in the first segment is β. Thus, if we offer the subset $S \subseteq N$ of products, then a customer purchases product $i \in S$ with probability $\beta \frac{v_i}{1+V(S)} + (1-\beta)\,\theta_i$. For notational brevity, throughout the chapter, we normalize the size of the first segment to one, in which case the size of the second segment relative to the first one is $\lambda = (1-\beta)/\beta$. Thus, if we offer the subset $S \subseteq N$ of products, then a customer purchases product $i \in S$ with the scaled probability $\frac{v_i}{1+V(S)} + \lambda\,\theta_i$. If a customer purchases product i, then we obtain a revenue of r_i. Our goal is to find a subset, or an assortment, to offer that maximizes the expected revenue from a customer, yielding the problem

$$\max_{S \subseteq N} \left\{ \sum_{i \in S} r_i \left(\frac{v_i}{1+V(S)} + \lambda\,\theta_i \right) \right\}. \tag{Mixture}$$

Since we normalize the size of the first segment to one, we can have $\frac{v_i}{1+V(S)} + \lambda\,\theta_i > 1$, but we can recover the purchase probabilities by scaling $\frac{v_i}{1+V(S)} + \lambda\,\theta_i$ for all $i \in N$ with β.

Working with such a mixture of the multinomial logit and independent demand models in the Mixture problem introduces nontrivial challenges. If we do not have the independent demand model in the mixture, then we can express the expected revenue under the multinomial logit model as a fraction, whose numerator and denominator are both linear functions, allowing us to use fractional programming techniques when solving the assortment optimization problem. We lose this fractional structure in the Mixture problem, but we will show that we can still solve this problem efficiently. Moreover, under only the multinomial logit model, there exists an optimal assortment that is nested by revenue, where we offer a certain number of products with the largest revenues. We lose the nested-by-revenue structure of the optimal solution in the Mixture problem. In Table 4.1, considering a problem instance with $n = 3$, $(r_1, r_2, r_3) = (50, 10, 5)$, $(v_1, v_2, v_3) = (0.5, 5, 0.01)$, $(\theta_1, \theta_2, \theta_3) = (0.05, 0.25, 0.7)$, and $\lambda = 1$, we show the expected revenue provided by each assortment. The optimal assortment is $\{1, 3\}$, which does not offer the product with second largest revenue, but offers the product with the smallest revenue. In this problem instance, noting that $\theta_3 = 0.7$, the customer segment with the independent demand model is interested in product 3 with a relatively large probability, so we offer product 3 to exploit this relatively large probability. Moreover, noting that $v_2 = 5$, the customer segment with the multinomial logit model associates a relatively large preference weight with product 2, but the revenue of product 2 is much smaller than that of product 1. Thus, product 2, if offered, attracts a significant fraction of the customer segment with the multinomial logit model while providing much smaller revenue than product 1, so we do not offer product 2. In the next section, we show that, roughly speaking, an optimal solution to the Mixture problem prioritizes product i when θ_i/v_i is larger, so a larger value for θ_i and a smaller value for v_i make product i more attractive to offer, which is consistent with the observation from Table 4.1.

Table 4.1: Expected revenue provided by all possible assortments

Assort.	Exp. Rev.	Assort.	Exp. Rev.
$\varnothing$	0	$\{1, 2\}$	16.54
$\{1\}$	19.17	$\{1, 3\}$	22.59
$\{2\}$	10.83	$\{2, 3\}$	14.33
$\{3\}$	3.55	$\{1, 2, 3\}$	20.03

Lastly, the multinomial logit model is a special case of the Markov chain choice model [30]. An example in Appendix C.1 shows that the mixture of multinomial logit and independent demand models is not a special case of the Markov chain choice model. Thus, existing results under the Markov chain choice model do not apply to our problem.

4.4 Assortment Optimization

In this section, we give an LP formulation for the Mixture problem and show that there exists an optimal solution that gives high priority to product i when θ_i/v_i is large. Using the decision variables x_0, $\boldsymbol{x} = \{x_i : i \in N\}$ and $\boldsymbol{y} = \{y_{ij} : i, j \in N\}$, we consider the LP

$$\max_{(x_0,\boldsymbol{x},\boldsymbol{y})\in\mathbb{R}\times\mathbb{R}_+^{n+n^2}} \left\{ \sum_{i\in N} r_i \left((v_i + \lambda\,\theta_i)\, x_i + \lambda\,\theta_i \sum_{j\in N} v_j\, y_{ij} \right) \ : \right. \qquad \text{(Assortment LP)}$$
$$x_0 + \sum_{i\in N} v_i\, x_i = 1,$$
$$x_i \le x_0 \quad \forall\, i \in N,$$
$$\left. y_{ij} \le x_i \quad \forall\, i,j \in N, \quad y_{ij} \le x_j \quad \forall\, i,j \in N \right\}.$$

Before showing that we can obtain an optimal solution to the Mixture problem by using the Assortment LP, we provide some intuition regarding the LP above. Using $\mathbf{1}(\cdot)$ to denote the indicator function, given a solution $\hat{S} \subseteq N$ to the Mixture problem, we construct a solution $(\hat{x}_0, \hat{\boldsymbol{x}}, \hat{\boldsymbol{y}})$ to the Assortment LP by setting $\hat{x}_0 = \frac{1}{1+V(\hat{S})}$, $\hat{x}_i = \mathbf{1}(i \in \hat{S})\, \hat{x}_0$, and $\hat{y}_{ij} = \mathbf{1}(i \in \hat{S},\ j \in \hat{S})\, \hat{x}_0$. Noting that $\sum_{i\in N} v_i\, \hat{x}_i = \hat{x}_0 \sum_{i\in N} v_i\, \mathbf{1}(i \in \hat{S}) = \hat{x}_0\, V(\hat{S})$, we have $\hat{x}_0 + \sum_{i\in N} v_i\, \hat{x}_i = \hat{x}_0\, (1 + V(\hat{S})) = 1$, so the solution $(\hat{x}_0, \hat{\boldsymbol{x}}, \hat{\boldsymbol{y}})$ satisfies the first constraint in the Assortment LP. Moreover, since $\mathbf{1}(i \in \hat{S}) \le 1$, $\mathbf{1}(i \in \hat{S},\ j \in \hat{S}) \le \mathbf{1}(i \in \hat{S})$, and $\mathbf{1}(i \in \hat{S},\ j \in \hat{S}) \le \mathbf{1}(j \in \hat{S})$, the solution $(\hat{x}_0, \hat{\boldsymbol{x}}, \hat{\boldsymbol{y}})$ satisfies the remaining constraints in the Assortment LP as well. Furthermore, for the Assortment LP, this solution provides an objective value of

$$\sum_{i\in N} r_i \left((v_i + \lambda\,\theta_i)\, \hat{x}_i + \lambda\,\theta_i \sum_{j\in N} v_j\, \hat{y}_{ij} \right)$$
$$= \sum_{i\in N} r_i \left((v_i + \lambda\,\theta_i)\, \mathbf{1}(i \in \hat{S}) + \lambda\,\theta_i \sum_{j\in N} v_j\, \mathbf{1}(i \in \hat{S},\ j \in \hat{S}) \right) \hat{x}_0$$

$$
\begin{aligned}
&= \sum_{i\in N} r_i \left(v_i + \lambda\,\theta_i + \lambda\,\theta_i \sum_{j\in N} v_j\,\mathbf{1}(j\in \hat{S}) \right) \mathbf{1}(i\in\hat{S})\,\hat{x}_0 \\
&= \sum_{i\in N} r_i \Big(v_i + \lambda\,\theta_i\,(1+V(\hat{S})) \Big)\,\mathbf{1}(i\in\hat{S})\,\hat{x}_0 \\
&= \sum_{i\in N} r_i \left(\frac{v_i}{1+V(\hat{S})} + \lambda\,\theta_i \right) \mathbf{1}(i\in\hat{S}),
\end{aligned}
$$

which is the objective function of the Mixture problem evaluated at $\hat{S}$. Thus, given a solution to the Mixture problem, we can construct a feasible solution to the Assortment LP, and the objective values of the two solutions match. To show that the Assortment LP is equivalent to the Mixture problem, we need to show the converse statement as well, which is what we do next. Note how collecting the terms $\lambda\,\theta_i\,\hat{x}_i$ and $\lambda\,\theta_i \sum_{j\in N} \hat{y}_{ij}$ above surprisingly yields the purchase probability of product i in the customer segment with the independent demand model.

To establish the converse statement, we build on the next lemma, which shows an important property of the basic feasible solutions to the Assortment LP.

Lemma 4.1 (Extreme Point Solutions). *Let $(\hat{x}_0, \hat{\boldsymbol{x}}, \hat{\boldsymbol{y}})$ be a basic feasible solution to the* Assortment LP. *Then, we have $\hat{x}_i \in \{0, \hat{x}_0\}$ for all $i \in N$.*

The proof of Lemma 4.1 follows by showing that if $\hat{x}_i \in (0, \hat{x}_0)$ for some $i \in N$, then we can perturb the solution $(\hat{x}_0, \hat{\boldsymbol{x}}, \hat{\boldsymbol{y}})$ to obtain two feasible solutions to the Assortment LP such that $(\hat{x}_0, \hat{\boldsymbol{x}}, \hat{\boldsymbol{y}})$ is a convex combination of the two feasible solutions. Note that perturbing $\hat{x}_i$ may require perturbing the other elements of the solution $(\hat{x}_0, \hat{\boldsymbol{x}}, \hat{\boldsymbol{y}})$ to ensure feasibility. We give the proof in Appendix C.2. In the next theorem, we use the above lemma to show that we can obtain an optimal solution to the Mixture problem by using an optimal solution to the Assortment LP.

Theorem 4.2 (LP Formulation). *For a basic optimal solution $(x_0^*, \boldsymbol{x}^*, \boldsymbol{y}^*)$ to the* Assortment LP, *let $S^* = \{i \in N : x_i^* > 0\}$. Then, S^* is an optimal solution to the* Mixture *problem.*

Proof. Let $\hat{S}$ be an optimal solution to the Mixture problem providing the optimal objective value $\hat{z}$, and let z^*_{LP} be the optimal objective value of the Assortment LP. By the discussion at the beginning of this section, given the solution $\hat{S}$ to the Mixture problem, we can construct a feasible solution to the Assortment LP with the objective value of $\hat{z}$. Therefore, we have $z^*_{\text{LP}} \geq \hat{z}$. On the other hand, by Lemma 4.1, we have $x^*_i = x^*_0$ for all $i \in S^*$ and $x^*_i = 0$ for all $i \in N \setminus S^*$. Since $(x^*_0, \boldsymbol{x}^*, \boldsymbol{y}^*)$ is a feasible solution to the Assortment LP, by the first constraint, we get $x^*_0 + \sum_{i \in S^*} v_i \, x^*_0 = 1$, so $x^*_0 = \frac{1}{1+V(S^*)} = x^*_i$ for all $i \in S^*$. In this case, by the last two constraints, we also get $y^*_{ij} \leq \frac{1}{1+V(S^*)}$ for all $i, j \in S^*$. If $i \notin S^*$ or $j \notin S^*$, then $x^*_i = 0$ or $x^*_j = 0$, so we have $y^*_{ij} = 0$. Let $Q^* = \{i \in S^* : r_i < 0\}$, and recall that z^*_{LP} is the optimal objective value of the Assortment LP and $\hat{z}$ is the optimal objective value of the Mixture problem. Since $x^*_i = 0$ when $i \notin S^*$ and $y^*_{ij} = 0$ when $i \notin S^*$ or $j \notin S^*$, evaluating the objective function of the Assortment LP at its optimal solution, we get

$$
\begin{aligned}
z^*_{\text{LP}} &= \sum_{i \in S^*} r_i \left((v_i + \lambda \, \theta_i) \, x^*_i + \lambda \, \theta_i \sum_{j \in S^*} v_j \, y^*_{ij} \right) \\
&= \sum_{i \in S^* \setminus Q^*} r_i \left((v_i + \lambda \, \theta_i) \, x^*_i + \lambda \, \theta_i \sum_{j \in S^*} v_j \, y^*_{ij} \right) + \sum_{i \in Q^*} r_i \left((v_i + \lambda \, \theta_i) \, x^*_i + \lambda \, \theta_i \sum_{j \in S^*} v_j \, y^*_{ij} \right) \\
&\overset{(a)}{\leq} \sum_{i \in S^* \setminus Q^*} r_i \left(\frac{v_i + \lambda \, \theta_i}{1 + V(S^*)} + \lambda \, \theta_i \sum_{j \in S^*} \frac{v_j}{1 + V(S^*)} \right) + \sum_{i \in Q^*} r_i \left((v_i + \lambda \, \theta_i) \, x^*_i + \lambda \, \theta_i \sum_{j \in S^*} v_j \, y^*_{ij} \right) \\
&= \sum_{i \in S^* \setminus Q^*} r_i \left(\frac{v_i}{1 + V(S^*)} + \lambda \, \theta_i \right) + \sum_{i \in Q^*} r_i \left((v_i + \lambda \, \theta_i) \, x^*_i + \lambda \, \theta_i \sum_{j \in S^*} v_j \, y^*_{ij} \right) \\
&\leq \sum_{i \in S^* \setminus Q^*} r_i \left(\frac{v_i}{1 + V(S^* \setminus Q^*)} + \lambda \, \theta_i \right) + \sum_{i \in Q^*} r_i \left((v_i + \lambda \, \theta_i) \, x^*_i + \lambda \, \theta_i \sum_{j \in S^*} v_j \, y^*_{ij} \right) \\
&\overset{(b)}{\leq} \hat{z} + \sum_{i \in Q^*} r_i \left((v_i + \lambda \, \theta_i) \, x^*_i + \lambda \, \theta_i \sum_{j \in S^*} v_j \, y^*_{ij} \right).
\end{aligned}
$$

Here, (a) holds since $r_i \geq 0$ and $x^*_i = \frac{1}{1+V(S^*)}$ for all $i \in S^* \setminus Q^*$ and $y^*_{ij} \leq \frac{1}{1+V(S^*)}$, whereas (b) holds since $S^* \setminus Q^*$ is a feasible but not necessarily an optimal solution to the Mixture problem.

Noting that $r_i < 0$ for all $i \in Q^*$, we have $\sum_{i \in Q^*} r_i \left((v_i + \lambda \, \theta_i) \, x^*_i + \lambda \, \theta_i \sum_{j \in S^*} v_j \, y^*_{ij} \right) \leq$

0. If $\sum_{i \in Q^*} r_i \, ((v_i + \lambda \, \theta_i) \, x_i^* + \lambda \, \theta_i \sum_{j \in S^*} v_j \, y_{ij}^*) < 0$, then the above chain of inequalities yields $z_{\mathsf{LP}}^* < \hat{z}$, contradicting the fact that $z_{\mathsf{LP}}^* \geq \hat{z}$, which we established at the beginning of the proof. Therefore, we have $\sum_{i \in Q^*} r_i \, ((v_i + \lambda \, \theta_i) \, x_i^* + \lambda \, \theta_i \sum_{j \in S^*} v_j \, y_{ij}^*) = 0$, but since $x_i^* > 0$, $r_i^* < 0$ and $v_i + \lambda \, \theta_i > 0$ for all $i \in Q^*$, for the last equality to hold, we must have $Q^* = \varnothing$. Since $Q^* = \varnothing$, noting that $z_{\mathsf{LP}} \geq \hat{z}$, all the inequalities in the above chain of inequalities hold as equalities. In particular, since (b) holds as an equality and $Q^* = \varnothing$, the objective value provided by the solution $S^* \setminus Q^* = S^*$ for the Mixture problem is $\hat{z}$, so S^* is an optimal solution to the Mixture problem. $\square$

The proof of Theorem 4.2 also shows that the Mixture problem and the Assortment LP have the same optimal objective values. In the proof of the theorem, we do not assume that the revenues of the products are non-negative. We will build on Theorem 4.2 when examining network revenue management problems. In that setting, the revenues of the products will be adjusted by the opportunity costs of the capacities used by the products, in which case, the revenues of some of the products can be negative. Nevertheless, when we focus only on solving the Mixture problem, we can a priori drop from consideration all products with nonpositive revenues, since if we drop such products, then the expected revenue from any assortment stays at least as large.

Prioritization of Products in an Optimal Assortment:

We give a result to intuitively suggest that there exists an optimal solution to the Mixture problem that prioritizes product i when r_i is larger or when θ_i / v_i is larger. Besides providing insight into the structure of the optimal assortment, this result allows us to construct a combinatorial algorithm for solving the Mixture problem. We start by reformulating the Mixture problem. Define the decision variables $\boldsymbol{w} = \{w_i : i \in N\} \in \{0,1\}^n$, where $w_i = 1$ if and only if we offer product i. Using z^* to denote the optimal objective value of the Mixture problem, we write this problem as

$$z^* \;=\; \max_{\boldsymbol{w} \in \{0,1\}^n} \left\{ \sum_{i \in N} r_i \left(\frac{v_i}{1 + \sum_{j \in N} v_j \, w_j} + \lambda \, \theta_i \right) w_i \right\}$$

$$= \max_{\boldsymbol{w}\in\{0,1\}^n} \left\{ \frac{\sum_{i\in N} r_i\,(v_i + \lambda\,\theta_i)\,w_i + \Big(\sum_{i\in N} v_i\,w_i\Big)\Big(\sum_{i\in N} r_i\,\lambda\,\theta_i\,w_i\Big)}{1+\sum_{i\in N} v_i\,w_i} \right\}, \qquad (4.1)$$

where the second equality follows by arranging the terms. As a function of $\boldsymbol{w}$, let $G(\boldsymbol{w})$ be the expression in the numerator of the fraction on the right side of (4.1).

By (4.1), we have $z^* \geq \frac{G(\boldsymbol{w})}{1+\sum_{i\in N} v_i\,w_i}$ for all $\boldsymbol{w} \in \{0,1\}^n$, and the inequality holds as equality at the optimal solution $\boldsymbol{w}^*$ to the Mixture problem. Thus, for all $\boldsymbol{w} \in \{0,1\}^n$, we have

$$\begin{aligned} z^* &\geq G(\boldsymbol{w}) - z^* \sum_{i\in N} v_i\,w_i \\ &= \sum_{i\in N} r_i\,(v_i+\lambda\,\theta_i)\,w_i + \left(\sum_{i\in N} v_i\,w_i\right)\left(\sum_{i\in N} r_i\,\lambda\,\theta_i\,w_i\right) - z^* \sum_{i\in N} v_i\,w_i \\ &= \sum_{i\in N} v_i \left(r_i + \lambda\,r_i\,\frac{\theta_i}{v_i} - z^*\right) w_i + \left(\sum_{i\in N} v_i\,w_i\right)\left(\sum_{i\in N} r_i\,\lambda\,\theta_i\,w_i\right), \end{aligned}$$

where the first equality follows by using the definition of $G(\boldsymbol{w})$. Once again, the inequality above holds for all $\boldsymbol{w} \in \{0,1\}^n$, and it holds as an equality at the optimal solution $\boldsymbol{w}^*$ to the Mixture problem. Thus, an optimal solution to the Mixture problem is a maximizer of the expression on the right side above over all $\boldsymbol{w} \in \{0,1\}^n$. In other words, letting $F(\boldsymbol{w}) = \sum_{i\in N} v_i\,(r_i + \lambda\,r_i\,\frac{\theta_i}{v_i} - z^*)\,w_i + \big(\sum_{i\in N} v_i\,w_i\big)\,\big(\sum_{i\in N} r_i\,\lambda\,\theta_i\,w_i\big)$ capture the expression on the right side above as a function of $\boldsymbol{w}$, $\boldsymbol{w}^*$ is an optimal solution to the problem $\max_{\boldsymbol{w}\in\{0,1\}^n} F(\boldsymbol{w})$.

In the next theorem, we use the discussion in the previous paragraph to provide insight into the structure of an optimal solution to the Mixture problem.

Theorem 4.3 (Prioritization of Products)**.** *There exists an optimal solution S^* to the* Mixture *problem that satisfies*

$$\min_{i\in S^*} \left\{ r_i \left(1 + \lambda\,\frac{\theta_i}{v_i}\,(1+V(S^*))\right)\right\} > \max_{i\in N\setminus S^*} \left\{ r_i \left(1 + \lambda\,\frac{\theta_i}{v_i}\,(1+V(S^*))\right)\right\}.$$

Proof. By the discussion preceding the theorem, letting $\boldsymbol{w}^*$ be an optimal solution to problem (4.1), we have $\boldsymbol{w}^* = \arg\max_{\boldsymbol{w}\in\{0,1\}^n} F(\boldsymbol{w})$. If we drop all products with nonpositive revenues from an assortment, then the expected revenue from the assortment stays at least as large, so we focus on the case where $r_i > 0$ for all $i \in N$. For each $i \in N$, the only term in $F(\boldsymbol{w})$ that depends on w_i in a nonlinear fashion is $v_i\, r_i\, \lambda\, \theta_i\, w_i^2$. Thus, $F(\boldsymbol{w})$ is directionally convex. In this case, we can relax the binary constraints to get $\boldsymbol{w}^* = \arg\max_{\boldsymbol{w}\in[0,1]^n} F(\boldsymbol{w})$. Let $V^* = \sum_{i\in N} v_i\, w_i^*$ and $\Theta^* = \sum_{i\in N} r_i\, \lambda\, \theta_i\, w_i^*$ for notational brevity. Differentiating $F(\boldsymbol{w})$ by using its definition, we get

$$\begin{aligned} \frac{\partial F(\boldsymbol{w})}{\partial w_i}\Big|_{\boldsymbol{w}=\boldsymbol{w}^*} &= v_i\left(r_i + \lambda\, r_i\, \frac{\theta_i}{v_i} - z^*\right) + v_i\left(\sum_{j\in N} r_j\, \lambda\, \theta_j\, w_j^*\right) + \left(\sum_{j\in N} v_j\, w_j^*\right) r_i\, \lambda\, \theta_i \\ &= v_i\left(r_i\left(1 + \lambda\, \frac{\theta_i}{v_i}\,(1+V^*)\right) - z^* + \Theta^*\right). \end{aligned}$$

We use $f_i(\boldsymbol{w}^*)$ to denote the derivative above. To show the result by contradiction, assume that $w_k^* = 1$ and $w_\ell^* = 0$ for some $k, \ell \in N$, and $r_k(1+\lambda\, \frac{\theta_k}{v_k}\,(1+V^*)) \le r_\ell(1+\lambda\, \frac{\theta_\ell}{v_\ell}\,(1+V^*))$.

Since $\boldsymbol{w}^* = \arg\max_{\boldsymbol{w}\in[0,1]^n} F(\boldsymbol{w})$ and $w_k^* = 1$, we have $f_k(\boldsymbol{w}^*) \ge 0$. Otherwise, a small decrease in w_k strictly increases the value of $F(\boldsymbol{w}^*)$. Similarly, we have $f_\ell(\boldsymbol{w}^*) \le 0$. Thus, we obtain

$$\begin{aligned} \frac{f_\ell(\boldsymbol{w}^*)}{v_\ell} &= r_\ell\left(1 + \lambda\, \frac{\theta_\ell}{v_\ell}\,(1+V^*)\right) - z^* + \Theta^* \\ &\qquad\qquad \le\ 0\ \le\ r_k\left(1 + \lambda\, \frac{\theta_k}{v_k}\,(1+V^*)\right) - z^* + \Theta^* = \frac{f_k(\boldsymbol{w}^*)}{v_k}. \end{aligned}$$

In this case, noting that $r_k(1 + \lambda\, \frac{\theta_k}{v_k}\,(1+V^*)) \le r_\ell(1+\lambda\, \frac{\theta_\ell}{v_\ell}\,(1+V^*))$, all the inequalities above must hold as equalities. In particular, we have $r_\ell(1+\lambda\, \frac{\theta_\ell}{v_\ell}\,(1+V^*)) - z^* + \Theta^* = 0$.

Define the solution $\hat{\boldsymbol{w}} = \{\hat{w}_i : i \in N\}$ as $\hat{w}_i = w_i^*$ for all $i \in N \setminus \{\ell\}$ and $\hat{w}_\ell = 1$.

Using the fact that the solutions $\hat{\boldsymbol{w}}$ and $\boldsymbol{w}^*$ differ only in the decision variable w_ℓ, we have

$$
\begin{aligned}
F(\hat{\boldsymbol{w}}) - F(\boldsymbol{w}^*) \;&=\; v_\ell \left(r_\ell + \lambda\, r_\ell \frac{\theta_\ell}{v_\ell} - z^* \right) + (V^* + v_\ell)(\Theta^* + r_\ell\, \lambda\, \theta_\ell) - V^*\, \Theta^* \\
&=\; v_\ell \left(r_\ell \left(1 + \lambda \frac{\theta_\ell}{v_\ell}\, (1 + V^*) \right) - z^* + \Theta^* \right) + v_\ell\, r_\ell\, \lambda\, \theta_\ell \\
&\overset{(a)}{=}\; v_\ell\, r_\ell\, \lambda\, \theta_\ell \;>\; 0,
\end{aligned}
$$

where (a) holds since $r_\ell(1 + \lambda \frac{\theta_\ell}{v_\ell}\,(1 + V^*)) - z^* + \Theta^* = 0$. Having $F(\hat{\boldsymbol{w}}) - F(\boldsymbol{w}^*) > 0$ contradicts the fact that $\boldsymbol{w}^*$ is an optimal solution to the problem $\max_{\boldsymbol{w} \in [0,1]^n} F(\boldsymbol{w})$. $\square$

By the theorem above, if product i has a larger value for r_i or θ_i / v_i, then the optimal assortment prioritizes this product. Besides providing insight into the structure of the optimal assortment, we can use Theorem 4.3 to construct a combinatorial algorithm for solving the Mixture problem. If we knew the value of $V(S^*)$, then letting $t = V(S^*)$, we could index the products such that $r_1(1 + \lambda \frac{\theta_1}{v_1}\,(1 + t)) \geq r_2(1 + \lambda \frac{\theta_2}{v_2}\,(1 + t)) \geq \ldots \geq r_n(1 + \lambda \frac{\theta_n}{v_n}\,(1 + t))$, in which case an optimal assortment would be of the form $\{1, \ldots, i\}$ for some $i \in N$. Thus, we would obtain an optimal assortment by checking the expected revenue from $O(n)$ candidate assortments, each of which is of the form $\{1, \ldots, i\}$ for some $i \in N$. To deal with the fact that we do not know the value of $V(S^*)$, we adopt an approach from [23]. Note that $g_i(t) = r_i(1 + \lambda \frac{\theta_i}{v_i}\,(1 + t))$ is a linear function of t. The n lines $\{g_i(\cdot) : i \in N\}$ intersect at $O(n^2)$ points. Let $t^1 \leq t^2 \leq \ldots \leq t^K$ with $K = O(n^2)$ be the intersection points of the n lines $\{g_i(\cdot) : i \in N\}$. That is, for each $k = 1, \ldots, K$, we have $g_i(t^k) = g_j(t^k)$ for some $i, j \in N$. Letting $t^0 = 0$ and $t^{K+1} = \infty$ for notational uniformity, for each $k = 0, \ldots, K$, if t takes values in the interval $[t^k, t^{k+1})$, then the ordering between the values $\{g_i(t) : i \in N\}$ does not change. To capture this ordering, we let the permutation $(\sigma_1^k, \ldots, \sigma_n^k) \in N^n$ be such that $g_{\sigma_1^k}(t) \geq g_{\sigma_2^k}(t) \geq \ldots \geq g_{\sigma_n^k}(t)$ for all $t \in [t^k, t^{k+1})$. In this case, we obtain the optimal assortment by checking the expected revenue from $O(n^3)$ candidate assortments, each of which is of the form $\{\sigma_1^k, \ldots, \sigma_i^k\}$ for some $i \in N$, $k = 0, \ldots, K$.

Thus, we can solve the Mixture problem without using an LP, but the Assortment LP becomes critical when we work with network revenue management problems.

4.5 Network Revenue Management

In the network revenue management setting, we have a set of resources, each with limited capacity. There is a finite number of time periods in the selling horizon. We choose the assortment of products to offer at each time period. The customer arriving at a time period chooses among the products according to a choice model. If the customer purchases a product, then we consume the capacities of a combination of resources and generate a revenue, both of which depend on the purchased product. The goal is to find a policy for choosing the assortment of products to offer at each time period to maximize the total expected revenue over the selling horizon. In [16] and [17], respectively, the authors formulate an LP approximation for the network revenue management problem. In their LP approximation, we have one decision variable for each assortment of products that we can offer to the customers, capturing the frequency with which we offer each assortment. Thus, the number of decision variables increases exponentially with the number of products, and it may be computationally cumbersome to solve the LP approximation.

In this section, we show that if the customers choose according to a mixture of multinomial logit and independent demand models, then we can formulate an equivalent LP whose number of decision variables and constraints increases only quadratically with the number of products. The optimal objective values of the two formulations are equal, and we can recover an optimal solution to one formulation by using an optimal solution to the other. To pin down the network revenue management problem, let T be the number of time periods in the selling horizon. At each time period, we have one customer arrival. The set of resources is $M = \{1, \ldots, m\}$. The capacity of resource q is c_q. The set of products is $N = \{1, \ldots, n\}$. If we offer the assortment $S \subseteq N$ of products, then a customer purchases product $i \in S$ with probability $\frac{v_i}{1+V(S)} + \lambda\, \theta_i$, in which case we generate a revenue of r_i and consume a_{qi} units of the capacity of resource q. We use the decision variables $\boldsymbol{w} = \{\boldsymbol{w}(S) : S \subseteq N\}$, where $w(S)$ is the probability that we offer assortment S at a time period. The LP approximation for the network revenue management problem is

$$\max_{\boldsymbol{w} \in \mathbb{R}_+^{2^n}} \left\{ T \sum_{S \subseteq N} \sum_{i \in S} r_i \left(\frac{v_i}{1 + V(S)} + \lambda\, \theta_i \right) w(S) \;\; : \right. \qquad \text{(Choice-Based LP)}$$

$$T \sum_{S \subseteq N} \sum_{i \in S} a_{qi} \left(\frac{v_i}{1 + V(S)} + \lambda\, \theta_i \right) w(S) \leq c_q \quad \forall\, q \in M,$$

$$\left. \sum_{S \subseteq N} w(S) = 1 \right\}.$$

Noting that $\sum_{i \in S} r_i (\frac{v_i}{v_0 + V(S)} + \lambda\, \theta_i)$ is the expected revenue at a time period at which we offer assortment S, the objective function above is the total expected revenue over the selling horizon. The first constraint ensures that the total expected capacity consumption of a resource does not exceed its capacity. The second constraint ensures that we offer an assortment at each time period.

To give an equivalent formulation for the Choice-Based LP, using the decision variables $(x_0, \boldsymbol{x}, \boldsymbol{y}) \in \mathbb{R} \times \mathbb{R}_+^{n+n^2}$ as in the Assortment LP, we consider the problem

$$\max_{(x_0, \boldsymbol{x}, \boldsymbol{y}) \in \mathbb{R} \times \mathbb{R}_+^{n+n^2}} \left\{ T \sum_{i \in N} r_i \left((v_i + \lambda\, \theta_i)\, x_i + \lambda\, \theta_i \sum_{j \in N} v_j\, y_{ij} \right) \;\; : \right. \qquad \text{(Compact LP)}$$

$$T \sum_{i \in N} a_{qi} \left((v_i + \lambda\, \theta_i)\, x_i + \lambda\, \theta_i \sum_{j \in N} v_j\, y_{ij} \right) \leq c_q \;\; \forall\, q \in M,$$

$$x_0 + \sum_{i \in N} v_i\, x_i = 1,$$

$$x_i \leq x_0 \;\; \forall\, i \in N,$$

$$\left. y_{ij} \leq x_i \;\; \forall\, i, j \in N, \;\; y_{ij} \leq x_j \;\; \forall\, i, j \in N \right\}.$$

The objective function of the Compact LP is the total expected revenue over the selling horizon. It turns out that we can use the expression $(v_i + \lambda\, \theta_i)\, x_i + \lambda\, \theta_i \sum_{j \in N} v_j\, y_{ij}$ to capture the expected number of purchases for product i at a time period. Remarkably, we can capture the expected number of purchases for the products by associating one decision variable with each product, as well as with each pair of products, rather than by associating one decision variable with each assortment of products. The first constraint ensures that the total expected capacity consumption of a resource does not exceed its capacity. The remaining constraints ensure that the choices of the customers are governed by our mixture choice model. We have 2^n decision variables and $m+1$ constraints in the Choice-Based LP, but n^2+n+1 decision variables and $2n^2+n+m+1$ constraints in the Compact LP. While solving the Choice-Based LP almost always requires using column generation, we may directly solve the Compact LP without using column generation.

There are heuristic policies that use an optimal primal or dual solution to the Choice-Based LP to decide which assortment of products to offer at each time period. In a randomized offer policy, letting $\boldsymbol{w}^*$ be an optimal solution to the Choice-Based LP, we offer assortment S with probability $w^*(S)$, after adjusting the offered assortment to accommodate the availabilities of the resources [54]. On the other hand, in a bid-price policy, letting $\boldsymbol{\mu}^* = \{\mu_q^* : q \in M\}$ be the optimal values of the dual variables associated with the first constraint in the Choice-Based LP, we use μ_q^* to capture the opportunity cost of a unit of resource q. If a customer purchases product i, then the opportunity cost of the resources used by product i is $\sum_{q\in M} a_{qi}\,\mu_q^*$, so the net expected revenue from the purchase is $r_i - \sum_{q\in M} a_{qi}\,\mu_q^*$. Therefore, the expected net revenue from offering assortment S is given by $\sum_{i\in S}(\frac{v_i}{1+V(S)} + \lambda\,\theta_i)(r_i - \sum_{q\in M} a_{qi}\,\mu_q^*)$, in which case we offer an assortment that maximizes this expected net revenue, once again after adjusting the assortment to accommodate the availabilities of the resources [19]. The discussion in this paragraph indicates that it is important to recover both optimal primal and dual solutions to the Choice-Based LP by using the Compact LP. In the next theorem, we show that the optimal objective values of the Choice-Based LP and Compact LP are equal and we can recover an optimal dual solution to the former by using the latter. In the next section, we focus on recovering an optimal primal solution to the Choice-Based LP by using the Compact LP. It turns out that relating the primal solutions of the two formulations will require more work. Lastly, we note that the choice models that govern the choices of the customers at all time periods in the Choice-Based LP have the same parameters. If the choices of the customers at different time periods are governed by choice models with different parameters, then we need to work with the decision variables $\{w_t(S) : S \subseteq N,\ t = 1, \ldots, T\}$, where $w_t(S)$ is the probability that we offer assortment S at time period t. All of our results continue to hold in this case.

Theorem 4.4. *The optimal objective values of the* Choice-Based LP *and* Compact LP *are the same. Furthermore, the optimal values of the dual variables for the first constraint in the* Choice-Based LP *and* Compact LP *are the same.*

Proof. Note that Compact LP is feasible and bounded, since setting $x_0 = 1$, $x_i = 0$ for all $i \in N$ and $y_{ij} = 0$ for all $i, j \in N$ yields a feasible solution and all decision variables are bounded. The last four constraints in the Compact LP are equivalent to the four constraints in the Assortment LP. For notational brevity, we capture the polytope defined by these constraints as

$$\mathcal{P} = \Big\{(x_0, \boldsymbol{x}, \boldsymbol{y}) \in \mathbb{R} \times \mathbb{R}_+^{n+n^2} : \ x_0 + \sum_{i \in N} v_i\, x_i = 1,\ x_i \le x_0\ \forall\, i \in N,\\ y_{ij} \le \min\{x_i, x_j\}\ \forall\, i, j \in N\Big\}.$$

We construct the Lagrangian for the Compact LP by associating the dual multipliers $\boldsymbol{\mu} = \{\mu_q : q \in M\}$ with the first constraint and relaxing this constraint, so the Lagrangian is

$$L(x_0, \boldsymbol{x}, \boldsymbol{y}; \boldsymbol{\mu}) = \sum_{i \in N} T\, r_i \left((v_i + \lambda\, \theta_i)\, x_i + \lambda\, \theta_i \sum_{j \in N} v_j\, y_{ij} \right) \\ + \sum_{q \in M} \mu_q \left(c_q - \sum_{i \in N} T\, a_{qi} \left((v_i + \lambda\, \theta_i)\, x_i + \lambda\, \theta_i \sum_{j \in N} v_j\, y_{ij} \right) \right)$$

$$= \sum_{i \in N} T \left(r_i - \sum_{q \in M} a_{iq}\, \mu_q \right) \left((v_i + \lambda\, \theta_i)\, x_i + \lambda\, \theta_i \sum_{j \in N} v_j\, y_{ij} \right) + \sum_{q \in M} c_q\, \mu_q.$$

In this case, using $D(\boldsymbol{\mu})$ to denote the dual function for the Compact LP as a function of the dual multipliers $\boldsymbol{\mu}$, we have $D(\boldsymbol{\mu}) = \max_{(x_0, \boldsymbol{x}, \boldsymbol{y}) \in \mathcal{P}} L(x_0, \boldsymbol{x}, \boldsymbol{y}; \boldsymbol{\mu})$.

The Compact LP is feasible and bounded, so strong duality holds. Thus, we can obtain the optimal objective value of the Compact LP by solving the dual problem $\min_{\boldsymbol{\mu} \in \mathbb{R}^m_+} D(\boldsymbol{\mu})$, and an optimal solution to the last problem gives the optimal values of the dual variables for the first constraint in the Compact LP. We write the dual function as

$$\begin{aligned}
D(\boldsymbol{\mu}) &= \max_{(x_0, \boldsymbol{x}, \boldsymbol{y}) \in \mathcal{P}} \left\{ \sum_{i \in N} T \left(r_i - \sum_{q \in M} a_{iq}\, \mu_q \right) \left((v_i + \lambda\, \theta_i)\, x_i + \lambda\, \theta_i \sum_{j \in N} v_j\, y_{ij} \right) \right\} + \sum_{q \in M} \mu_q\, c_q \\
&\overset{(a)}{=} \max_{S \subseteq N} \left\{ \sum_{i \in S} T \left(r_i - \sum_{q \in M} a_{iq}\, \mu_q \right) \left(\frac{v_i}{1 + V(S)} + \lambda\, \theta_i \right) \right\} + \sum_{q \in M} \mu_q\, c_q \\
&\overset{(b)}{=} \max_{\boldsymbol{w} \in \mathbb{R}^{2^n}_+} \left\{ \sum_{S \subseteq N} \sum_{i \in S} T \left(r_i - \sum_{q \in M} a_{iq}\, \mu_q \right) \left(\frac{v_i}{1 + V(S)} + \lambda\, \theta_i \right) w(s) : \sum_{S \subseteq N} w(s) = 1 \right\} \\
&\qquad\qquad + \sum_{q \in M} \mu_q\, c_q \\
&= \max_{\boldsymbol{w} \in \mathbb{R}^{2^n}_+} \left\{ \sum_{S \subseteq N} \sum_{i \in S} T\, r_i \left(\frac{v_i}{1 + V(S)} + \lambda\, \theta_i \right) w(s) \right. \\
&\qquad\qquad \left. + \sum_{q \in M} \mu_q \left(c_q - \sum_{S \subseteq N} \sum_{i \in S} T\, a_{qi} \left(\frac{v_i}{1 + V(S)} + \lambda\, \theta_i \right) w(s) \right) : \sum_{S \subseteq N} w(s) = 1 \right\}.
\end{aligned}$$

In the chain of equalities above, (a) uses the fact that the LP on the left side of this equality is equivalent to the Assortment LP after replacing the revenue of product i with $r_i - \sum_{q \in M} a_{qi}\, \mu_q$. Thus, we can obtain the optimal objective value of this LP by solving the Mixture problem after replacing the revenue of product i with $r_i - \sum_{q \in M} a_{qi}\, \mu_q$. On the other hand, (b) holds, since picking one assortment that maximizes the expected revenue in the Mixture problem is equivalent to randomizing over all possible assortments. The optimal objective value of the last LP above gives the dual function for the Choice-Based LP. Therefore, the Compact LP and Choice-Based LP have the same dual functions. In this case, if we minimize the dual functions for the two LP formulations over all $\boldsymbol{\mu} \in \mathbb{R}^m_+$, then we get the same optimal objective value, so the two LP formulations have the same optimal objective value. Furthermore, the minimizers of the dual functions for the two LP formulations must be the same, which implies that the optimal values of the dual variables for the first constraint in the two LP formulations are the same. $\square$

By Theorem 4.4, we can solve the Compact LP to obtain the optimal dual variables of the first constraint in the Choice-Based LP, in which case we can use these dual variables to implement the bid-price policy. Considering the LP on the left side of (a) in the proof of the theorem, we do not a priori know whether product i satisfies $r_i - \sum_{q \in M} a_{qi}\, \mu_q \geq 0$. Therefore, as discussed immediately after the proof of Theorem 4.2, it is important that the Assortment LP can recover the optimal objective value of the Mixture problem even when some of the products have nonpositive revenues. To close this section, we iterate that we can have $\frac{v_i}{v_0 + V(S)} + \lambda\, \theta_i > 1$, since we normalize the size of the customer segment with the multinomial logit model to one. To recover the purchase probability of product i, we need to scale $\frac{v_i}{v_0 + V(S)} + \lambda\, \theta_i$ with β. Thus, the purchase probabilities in the Choice-Based LP and Compact LP have implicitly been scaled with $1/\beta$, so the resource capacities in these LP formulations have implicitly been scaled with $1/\beta$ as well.

4.6 Recovering a Primal Solution

We focus on recovering an optimal primal solution to the Choice-Based LP by using the Compact LP. Throughout this section, we follow the convention that if the Compact LP has multiple optimal solutions, then we pick any one that has the largest value for the decision variable x_0. It is simple to implement this convention in practice. In particular, for $\epsilon > 0$, we can add the additional term $\epsilon\, x_0$ to the objective function of the Compact LP. If ϵ is small enough, then the additional term favors an optimal solution with the largest value of x_0. If it is not clear how small ϵ should be, then another approach would be to first solve the Compact LP and obtain its optimal objective value. Letting z^*_{LP} be the optimal objective value of the Compact LP, we can then solve another LP where we maximize x_0 in the objective function, subject to the constraint that

$T\sum_{i\in N} r_i\,((v_i + \lambda\,\theta_i)\,x_i + \lambda\,\theta_i \sum_{j\in N} v_j\,y_{ij}) \geq z^*_{\text{LP}}$, along with all constraints in the Compact LP. In this case, we get a solution with the largest value for the decision variable x_0, providing an objective value of at least z^*_{LP}, so it must be optimal. In the next lemma, we establish a useful property of the basic optimal solutions to the Compact LP. The proof is given in Appendix C.3.

Lemma 4.5 (Extreme Point Optimal Solutions)**.** *Let $(x^*_0, \boldsymbol{x}^*, \boldsymbol{y}^*)$ be a basic optimal solution to the* Compact LP*. Then, we have $y^*_{ij} = \min\{x^*_i, x^*_j\}$ for all $i, j \in N$.*

The proof, which is nontrivial, explicitly uses the fact that we pick a basic optimal solution that has the largest value for the decision variable x_0. We can generate examples to show that we may not have $y^*_{ij} = \min\{x^*_i, x^*_j\}$ for other basic optimal solutions. Also, note that the last two constraints in the Compact LP do not immediately imply that $y^*_{ij} = \min\{x^*_i, x^*_j\}$, since the first constraint in this LP may not allow setting $y^*_{ij} = \min\{x^*_i, x^*_j\}$ in a feasible solution to the Compact LP. Next, we focus on the main result of this section and give a remarkably efficient approach for recovering an optimal primal solution to the Choice-Based LP from an optimal solution to the Compact LP.

Let $(x^*_0, \boldsymbol{x}^*, \boldsymbol{y}^*)$ be a basic optimal solution to the Compact LP. We index the products so that $x^*_1 \geq x^*_2 \geq \ldots \geq x^*_n$. Defining the set $S_i = \{1, \ldots, i\}$ with $S_0 = \varnothing$, for each $i = 0, 1, \ldots, n$, we set

$$\hat{w}(S_i) = (x^*_i - x^*_{i+1})\,(1 + V(S_i)), \qquad \text{(Recovery)}$$

where we follow the convention that $x^*_{n+1} = 0$. Noting that $x^*_0 \geq x^*_i$ for all $i \in N$ by the third constraint in the Compact LP, we have $\hat{w}(S_0) = x^*_0 - x^*_1 \geq 0$.

We define the solution $\hat{\boldsymbol{w}}$ to the Choice-Based LP as follows. For all $i = 0, 1, \ldots, n$, we set $\hat{w}(S_i)$ as in the Recovery formula. For $S \notin \{S_0, S_1, \ldots, S_n\}$, we set $\hat{w}(S) = 0$.

In the next theorem, we show that the solution $\hat{\boldsymbol{w}}$ that we construct by using the Recovery formula as discussed above is an optimal solution to the Choice-Based LP.

Theorem 4.6 (Recovering an Optimal Solution)**.** *For a basic optimal solution* $(x_0^*, \boldsymbol{x}^*, \boldsymbol{y}^*)$ *to the* Compact LP*, let* $\hat{\boldsymbol{w}} = \{\hat{w}(S) : S \subseteq N\}$ *be constructed as in the* Recovery *formula with* $\hat{w}(S) = 0$ *for all* $S \notin \{S_0, S_1, \ldots, S_n\}$*. Then,* $\hat{\boldsymbol{w}}$ *is an optimal solution to the* Choice-Based LP.

Proof. For notational brevity, let $\Lambda_i^* = T\,(v_i + \lambda\,\theta_i)\,x_i^* + T\,\lambda\,\theta_i \sum_{j \in N} v_j\, y_{ij}^*$. By Lemma 4.4, the optimal objective values of the Compact LP and Choice-Based LP are equal. Let z_{LP}^* be their common optimal objective value. Noting the objective function of the Compact LP, we have $z_{\text{LP}}^* = \sum_{i \in N} r_i\, \Lambda_i^*$. Furthermore, by the first constraint in the Compact LP, we have $\sum_{i \in N} a_{qi}\, \Lambda_i^* \le c_q$. We will show that $\sum_{S \subseteq N} \hat{w}(S) = 1$ and $\sum_{S \subseteq N} \mathbf{1}(i \in S)\,\big(\frac{v_i}{1+V(S)} + \lambda\,\theta_i\big)\,\hat{w}(S) = \Lambda_i^*$. In this case, we get

$$\begin{aligned}\sum_{S \subseteq N} \sum_{i \in S} r_i \left(\frac{v_i}{v_0 + V(S)} + \lambda\,\theta_i \right) \hat{w}(S) &= \sum_{i \in N} \sum_{S \subseteq N} r_i\, \mathbf{1}(i \in S) \left(\frac{v_i}{v_0 + V(S)} + \lambda\,\theta_i \right) \hat{w}(S) \\ &= \sum_{i \in N} r_i\, \Lambda_i^* = \; z_{\text{LP}}^*.\end{aligned}$$

Thus, the solution $\hat{\boldsymbol{w}}$ provides an objective value of z_{LP}^* for the Choice-Based LP, which is the optimal objective value of this LP. Moreover, replacing r_i with a_{qi} in the chain of equalities above and carrying out the same computation, we get $\sum_{S \subseteq N} \sum_{i \in S} a_{qi} \big(\frac{v_i}{v_0 + V(S)} + \lambda\,\theta_i\big)\,\hat{w}(S) = \sum_{i \in N} a_{qi}\, \Lambda_i^*$. So, since $\sum_{i \in N} a_{qi}\, \Lambda_i^* \le c_q$, $\hat{\boldsymbol{w}}$ is a feasible solution to the Choice-Based LP. Noting that $\hat{\boldsymbol{w}}$ provides an objective value of z_{LP}^* for this LP, $\hat{\boldsymbol{w}}$ is an optimal solution for the Choice-Based LP, as desired.

First, we show that $\sum_{S \subseteq N} \hat{w}(S) = 1$. By the definition of S_i, we have $V(S_i) - V(S_{i-1}) = v_i$ for all $i = 1, \ldots, n$. Thus, using the Recovery formula, we get

$$\begin{aligned}\sum_{S \subseteq N} \hat{w}(S) \;&\overset{(a)}{=}\; \sum_{i=0}^{n} \hat{w}(S_i) \;=\; \sum_{i=0}^{n} (x_i^* - x_{i+1}^*)\,(1 + V(S_i)) \\ &=\; \sum_{i=0}^{n} x_i^*\,(1 + V(S_i)) - \sum_{i=0}^{n} x_{i+1}^*\,(1 + V(S_i))\end{aligned}$$

$$
\begin{aligned}
&= \left(x_0^* \,(1 + V(S_0)) + \sum_{i=1}^{n} x_i^* \,(1 + V(S_i)) \right) \\
&\qquad\qquad - \left(\sum_{i=1}^{n} x_i^* \,(1 + V(S_{i-1})) + x_{n+1}^* \,(1 + V(S_n)) \right) \\
&\overset{(b)}{=} \; x_0^* + \sum_{i=1}^{n} x_i^* \,(V(S_i) - V(S_{i-1})) \; = \; x_0^* + \sum_{i=1}^{n} v_i \, x_i^* \; \overset{(c)}{=} \; 1,
\end{aligned}
$$

where (a) holds since $\hat{w}(S) = 0$ for all $S \notin \{S_0, S_1, \ldots, S_n\}$, (b) holds since $x_{n+1}^* = 0$, and (c) holds since the solution $(x_0^*, \boldsymbol{x}^*, \boldsymbol{y}^*)$ satisfies the second constraint in the Compact LP.

Second, we show that $\sum_{S \subseteq N} \mathbf{1}(i \in S) \left(\frac{v_i}{1+V(S)} + \lambda \, \theta_i \right) \hat{w}(S) = \Lambda_i^*$. Noting the definition of $\hat{\boldsymbol{w}}$ in the Recovery formula, for each $k = 0, 1, \ldots, n$, we have

$$
\begin{aligned}
\hat{w}(S_k) \; &= \; (x_k^* - x_{k+1}^*) \,(1 + V(S_k)) \; = \; x_k^* - x_{k+1}^* + (x_k^* - x_{k+1}^*) \, V(S_k) \\
&= \; x_k^* - x_{k+1}^* + \sum_{\ell \in N} \mathbf{1}(\ell \leq k) \,(x_k^* - x_{k+1}^*) \, v_\ell,
\end{aligned}
$$

where the last equality uses the fact that $S_i = \{1, \ldots, i\}$ and $V(S) = \sum_{i \in S} v_i$. By Lemma 4.5, we have $y_{ij}^* = \min\{x_i^*, x_j^*\}$ for all $i, j \in N$. Thus, since we index the products such that $x_1^* \geq x_2^* \geq \ldots \geq x_n^*$, we have $y_{ij}^* = x_i^*$ for $i \geq j$ and $y_{ij}^* = x_j^*$ for $i < j$. In other words, letting $a \vee b = \max\{a, b\}$, we have $y_{ij}^* = x_{i \vee j}^*$. Using the last chain of equalities displayed above, for each $i \in N$, we get

$$
\begin{aligned}
&\sum_{k \in N} \mathbf{1}(k \geq i) \, \hat{w}(S_k) \\
&= \; \sum_{k \in N} \mathbf{1}(k \geq i) \,(x_k^* - x_{k+1}^*) + \sum_{k \in N} \sum_{\ell \in N} \mathbf{1}(k \geq i) \, \mathbf{1}(\ell \leq k) \,(x_k^* - x_{k+1}^*) \, v_\ell \\
&\overset{(d)}{=} \; \sum_{k \in N} \mathbf{1}(k \geq i) \,(x_k^* - x_{k+1}^*) + \sum_{\ell \in N} v_\ell \sum_{k \in N} \mathbf{1}(k \geq i \vee \ell) \,(x_k^* - x_{k+1}^*) \\
&\overset{(e)}{=} \; x_i^* + \sum_{\ell \in N} v_\ell \, x_{i \vee \ell}^* \; = \; x_i^* + \sum_{\ell \in N} v_\ell \, y_{i\ell}^*, \qquad (4.2)
\end{aligned}
$$

where (d) holds since $\mathbf{1}(k \geq i)\,\mathbf{1}(\ell \leq k) = 1$ if and only if $\mathbf{1}(k \geq i \vee \ell)$ and (e) holds by canceling the telescoping terms in the first and third sums on the left side of the equality.

By the Recovery formula, we have $\sum_{k\in N} \mathbf{1}(k \geq i) \frac{1}{1+V(S_k)}\, \hat{w}(S_k) = \sum_{k\in N} \mathbf{1}(k \geq i)\,(x_k^* - x_{k+1}^*) = x_i^*$. In this case, noting that $i \in S_k$ if and only if $k \geq i$, we obtain

$$\begin{aligned}
&\sum_{S\subseteq N} \mathbf{1}(i \in S) \left(\frac{v_i}{1+V(S)} + \lambda\,\theta_i \right) \hat{w}(S) \\
&= \sum_{k\in N} \mathbf{1}(i \in S_k) \left(\frac{v_i}{1+V(S_k)} + \lambda\,\theta_i \right) \hat{w}(S_k) \\
&= \sum_{k\in N} \mathbf{1}(k \geq i) \left(\frac{v_i}{1+V(S_k)} + \lambda\,\theta_i \right) \hat{w}(S_k) \\
&= v_i \sum_{k\in N} \mathbf{1}(k \geq i)\, \frac{1}{1+V(S_k)}\, \hat{w}(S_k) + \lambda\,\theta_i \sum_{k\in N} \mathbf{1}(k \geq i)\, \hat{w}(S_k) \\
&\overset{(f)}{=} v_i\, x_i^* + \lambda\,\theta_i \left(x_i^* + \sum_{\ell\in N} v_\ell\, y_{i\ell}^* \right) \overset{(g)}{=} \Lambda_i^*,
\end{aligned}$$

where (f) follows from (4.2) and (g) holds by the definition of Λ_i^*. Thus, the two identities that we claim to hold at the beginning of the proof indeed hold. $\square$

By the theorem above, noting the Recovery formula, after we solve the Compact LP, recovering an optimal solution to the Choice-Based LP requires simply sorting the values of the decision variables $\{x_i : i \in N\}$. Furthermore, the Recovery formula implies that there exists an optimal solution $\boldsymbol{w}^*$ to the Choice-Based LP such that $w^*(S) = 0$ for all $S \not\in \{S_0, S_1, \ldots, S_n\}$, so there exists an optimal solution to the Choice-Based LP that offers at most $n+1$ subsets. In this solution, since the sets $\{S_i : i = 0, 1, \ldots, n\}$ satisfy $S_0 \subseteq S_1 \subseteq \ldots \subseteq S_n$, if $w^*(S) > 0$ and $w^*(Q) > 0$, then we have either $S \supseteq Q$ or $Q \supseteq S$. Putting the last two observations together, there exists an optimal solution to the Choice-Based LP that offers at most $n+1$ assortments and these assortments are related to each other in the sense that one assortment is included in another one.

4.7 Numerical Experiments

We provide two sets of computational experiments. In the first set, we test the ability of the mixture of multinomial logit and independent demand models to capture the choice process of the customers. In the second set, we check the computational benefits of using the Compact LP, instead of solving the Choice-Based LP directly by using column generation.

4.7.1 Prediction Ability of the Mixture Model

In this section, we test the benefits of using the mixture of multinomial logit and independent demand models to predict the purchase behavior of the customers.

Experimental Setup:

We generate the past purchase history of customers under the assumption that the choices of the customers are governed by a complex ground choice model that is very different from the multinomial logit model. The past purchase history includes the assortment of products offered to each customer and the product, if any, purchased within the assortment. We split the purchase history into training and testing data. We fit a mixture of multinomial logit and independent demand models to the training data and test the performance of the fitted model on the testing data. As a benchmark, we also fit a pure multinomial logit model to the same training data. In all of our test problems, we have $n = 10$ products. In the ground choice model, we have $p = 50$ customer types. Indexing the customer types by $P = \{1, \ldots, p\}$, customer type ℓ is characterized by a preference list of products $\boldsymbol{\sigma}^\ell = (\sigma^\ell(1), \sigma^\ell(2), \ldots, \sigma^\ell(k^\ell))$, with $\sigma^\ell(i) \in N$ for all $i = 1, \ldots, k^\ell$, where $\sigma^\ell(i)$ is the i-th most preferred product by a customer of type ℓ and k^ℓ is the number of products in the preference list. A customer of type ℓ arrives into the system with probability β_ℓ. An arriving customer chooses her most preferred available product in her preference list. If no product in her preference list is available, then the customer leaves without a purchase.

The parameters of the ground choice model are the collection of preference lists $\{\boldsymbol{\sigma}^\ell : \ell \in P\}$ and the arrival probabilities $\{\beta^\ell : \ell \in P\}$. To generate these parameters, we consider the case where the products have an inherent ordering $1 \succ 2 \succ \ldots \succ n$, in which product 1 has the highest quality and highest price, whereas product n has the lowest quality and lowest price. A customer of a particular type has a maximum price she can afford and minimum quality she accepts. In this case, the customer generally chooses the highest-quality product that is available within this range, but we add some noise to introduce some deviation from the inherent ordering of the products. In particular, for each customer type ℓ, we randomly choose an interval of products $[i^\ell, j^\ell]$ with $i^\ell < j^\ell$, so that a customer of type ℓ cannot afford products with price higher than that of product i^ℓ and does not accept products with quality lower than that of product j^ℓ. Considering the ordered list $(\gamma^\ell(1), \gamma^\ell(2), \ldots, \gamma^\ell(k^\ell)) = (i^\ell, i^\ell+1, \ldots, j^\ell)$ with $k^\ell = j^\ell - i^\ell + 1$, we make 20 random swaps to change the position of a product with its successor so that the preference list only roughly follows the inherent ordering of the products. In this way, we obtain the preference list of products $(\sigma^\ell(1), \sigma^\ell(2), \ldots, \sigma^\ell(k^\ell))$ for customers of type ℓ. Following the approach described so far in this paragraph, we generate the preference lists for $p - n = 40$ customer types. The preference lists for the remaining $n = 10$ customer types is a singleton, each including one of the products, so the customers of these types are unwilling to substitute between the products. To come up with the arrival probabilities, sampling ζ^ℓ from the uniform distribution over [0,1], for all $\ell = 1, \ldots, p-n$, we set $\beta^\ell = (1-\Theta)\,\zeta^\ell / \sum_{k=1}^{p-n} \zeta^k$, whereas for all $\ell = p-n+1, \ldots, p$, we set $\beta^\ell = \Theta\,\zeta^\ell / \sum_{k=p-n+1}^{p} \zeta^k$, where $\Theta \in (0,1)$ is a parameter that we vary. In this case, we have $\sum_{\ell=p-n+1}^{p} \beta^\ell = \Theta$, so Θ fraction of the customers are unwilling to substitute between the products.

Once we generate the ground choice model as in the previous paragraph, we generate the past purchase histories of customers who choose according to the ground choice model. The past purchase history consists of the pairs $\{(S_t, i_t) : t = 1, \ldots, \tau\}$, where τ is the number of customers in the history, S_t is the assortment that we offer to customer t, and i_t

is the product that this customer chooses. If customer t does not purchase anything, then we have $i_t = 0$. To generate the assortment S_t, we include each product in the assortment S_t with probability $\rho \in (0, 1)$, where ρ is another parameter that we vary. We sample the product i_t among the products in S_t and the no-purchase option according to the ground choice model.

We use maximum likelihood estimation to fit a mixture of multinomial logit and independent demand models to the past purchase history. Similarly, we use maximum likelihood estimation to fit a pure multinomial logit model without having an independent demand model in the mixture.

Testing Prediction Performance:

Varying $(\Theta, \rho) \in \{0.2, 0.3, 0.4, 0.5\} \times \{0.4, 0.5, 0.6\}$, we obtain 12 parameter combinations. For each parameter combination, we generate the ground choice model as described earlier in this section. Using the ground choice model, we generate the past purchase history of τ customers. We vary $\tau \in \{1250, 2500, 5000\}$ to capture three levels of data availability in the training data that we use to fit the choice models. Following the same approach to generate the training data, we generate the past purchase history for another 10,000 customers to use as the testing data. For each combination of (Θ, ρ) and τ, we replicate our results 50 times to get a better understanding of how much they change from one replication to another. We regenerate the ground choice model in each of these replications. We compare the two fitted choice models in terms of the out-of-sample log-likelihoods and the deviation between the choice probabilities under each fitted choice model and the exact ground choice model. Throughout this section, we use MIX to refer to the fitted mixture of multinomial logit and independent demand models, whereas MNL to refer to the fitted pure multinomial logit model.

In Table 4.2, we compare MIX and MNL in terms of their out-of-sample log-likelihoods. In each of the 50 replications, after generating the ground choice model, we sample training data and testing data using the ground choice model. We fit MIX and MNL to the training data, and we compute the log-likelihood of the testing data under the fitted choice models. The first column gives the parameter configuration (Θ, ρ). In the rest of the table, there are three blocks, each with five columns. The three blocks correspond to the values of $\tau \in \{1250, 2500, 5000\}$, capturing three levels of data availability. In each block, the first column gives the average out-of-sample log-likelihood of MIX, where the average is computed over 50 replications. The second column gives the average out-of-sample log-likelihood of MNL. The third column gives the average percent gap between out-of-sample log-likelihoods of MIX and MNL. The fourth column gives the number of replications out of 50 in which the out-of-sample log-likelihood of MIX is better than that of MNL, whereas the fifth column gives the number of replications in which the outcome is reversed. Our results indicate that the out-of-sample log-likelihoods of MIX are noticeably larger than those of MNL. For the smallest training data availability with $\tau = 1250$, there are a few replications in which the out-of-sample log-likelihoods of MNL are larger than those of MIX, but for $\tau = 2500$ and $\tau = 5000$, the out-of-sample log-likelihoods of MIX are quite consistently larger than those of MNL.

Note that MIX has $2n$ parameters, whereas MNL has n parameters. Thus, MIX provides more flexibility for capturing the customer choice behavior. With its larger number of parameters, however, MIX may overfit to the training data, resulting in poor out-of-sample log-likelihoods, especially when we have too little training data. Thus, it is not guaranteed that the out-of-sample log-likelihoods of MIX will be larger than those of MNL. Nevertheless, overfitting does not seem to be a concern for MIX, and the out-of-sample log-likelihoods of MIX are larger than those of MNL in an overwhelming majority of our replications. The gap between the out-of-sample log-likelihoods of MIX and MNL becomes more pronounced as the amount of training data increases, corresponding to larger values for τ. Shortly, we demonstrate that such improvements in out-of-sample log-likelihoods translate into significant improvements in expected revenues.

Table 4.2: Out-of-sample log-likelihoods of the two fitted choice models

	$\tau = 1{,}250$					$\tau = 2{,}500$					$\tau = 5{,}000$				
Param. (Θ, ρ)	MIX Like.	MNL Like.	Perc. Gap	MIX Better	MNL Better	MIX Like.	MNL Like.	Perc. Gap	MIX Better	MNL Better	MIX Like.	MNL Like.	Perc. Gap	MIX Better	MNL Better
$(0.2, 0.4)$	-12,092	-12,195	0.85	46	4	-12,041	-12,167	1.05	50	0	-12,024	-12,158	1.12	50	0
$(0.2, 0.5)$	-13,753	-13,832	0.57	47	3	-13,710	-13,807	0.71	50	0	-13,690	-13,796	0.78	50	0
$(0.2, 0.6)$	-15,194	-15,250	0.37	45	5	-15,156	-15,224	0.45	49	1	-15,137	-15,214	0.51	50	0
$(0.3, 0.4)$	-12,510	-12,596	0.69	48	2	-12,461	-12,570	0.88	50	0	-12,441	-12,561	0.96	50	0
$(0.3, 0.5)$	-14,269	-14,345	0.54	49	1	-14,228	-14,321	0.66	50	0	-14,208	-14,310	0.72	50	0
$(0.3, 0.6)$	-15,841	-15,904	0.40	46	4	-15,804	-15,880	0.48	50	0	-15,784	-15,870	0.54	50	0
$(0.4, 0.4)$	-12,772	-12,867	0.74	49	1	-12,734	-12,847	0.88	50	0	-12,716	-12,838	0.96	50	0
$(0.4, 0.5)$	-14,624	-14,705	0.55	48	2	-14,588	-14,686	0.67	49	1	-14,569	-14,677	0.74	50	0
$(0.4, 0.6)$	-16,286	-16,358	0.44	48	2	-16,253	-16,341	0.54	50	0	-16,235	-16,332	0.60	50	0
$(0.5, 0.4)$	-12,881	-12,966	0.66	50	0	-12,845	-12,946	0.79	50	0	-12,826	-12,938	0.87	50	0
$(0.5, 0.5)$	-14,850	-14,932	0.55	50	0	-14,810	-14,912	0.69	50	0	-14,793	-14,903	0.74	50	0
$(0.5, 0.6)$	-16,605	-16,689	0.50	50	0	-16,567	-16,669	0.61	50	0	-16,549	-16,659	0.67	50	0
Average			0.57	48	2			0.70	49.83	0.17			0.77	50	0

Testing Revenue Performance:

In Table 4.3, we compare MIX and MNL in terms of their revenue performance. The layout of this table is identical to that of Table 4.2, with three blocks capturing three levels of data availability. In replication q, let $\phi_i^q(S)$ be the choice probability of product i within assortment S under the fitted MIX. We generate 100 samples of the product revenues, sampling the revenue of each product from the uniform distribution over $[1, 10]$. In replication q, letting $(r_1^{qk}, \ldots, r_n^{qk})$ be the revenues of the products in the k-th sample, we use $\hat{S}^{qk}$ to denote the optimal assortment to offer under the assumption that the customers choose under the fitted MIX. In other words, $\hat{S}^{qk}$ is an optimal solution to the problem $\max_{S \subseteq N} \sum_{i \in S} r_i^{qk}\, \phi_i^q(S)$. The customers, however, actually choose according to the ground choice model. In replication q, letting let $P_i^q(S)$ be the choice probability of product i within assortment S under the ground choice model, we compute the actual expected revenue from assortment $\hat{S}^{qk}$ as $R_{\text{MIX}}^{qk} = \sum_{i \in \hat{S}^{qk}} r_i^{qk}\, P_i^q(S)$. Averaging over all the 100 revenue samples, in replication q, we capture the expected revenue performance of the fitted MIX by $\text{Rev}_{\text{MIX}}^q = \frac{1}{100} \sum_{k=1}^{100} R_{\text{MIX}}^{qk}$. In each block, considering the 50 replications, the first column gives the average of $\{\text{Rev}_{\text{MIX}}^q : q = 1, \ldots, 50\}$. The second column gives the average of $\{\text{Rev}_{\text{MNL}}^q : q = 1, \ldots, 50\}$, where $\text{Rev}_{\text{MNL}}^q$ captures the expected revenue performance of the fitted MNL in replication q, computed in a fashion similar to $\text{Rev}_{\text{MIX}}^q$. The third column gives the percent gap between the first two columns. The fourth column gives the number of replications in which the expected revenue performance of MIX is better than that of MNL, whereas the fifth column gives the number of replications in which the outcome is reversed.

Our results indicate that fitting MIX to the training data can provide assortments with significantly larger revenues, when compared to fitting MNL to the training data. The improvements in the expected revenue provided by MIX are consistent over an overwhelming majority of our replications and can exceed 6%. As τ gets larger and the amount of training data increases, the improvements in the expected revenue by MIX become more noticeable.

Table 4.3: Expected revenues obtained by using the two fitted choice models

Param. (Θ, ρ)	$\tau = 1{,}250$ MIX Rev.	MNL Rev.	Perc. Gap	MIX Better	MNL Better	$\tau = 2{,}500$ MIX Rev.	MNL Rev.	Perc. Gap	MIX Better	MNL Better	$\tau = 5{,}000$ MIX Rev.	MNL Rev.	Perc. Gap	MIX Better	MNL Better
$(0.2, 0.4)$	5.79	5.67	2.04	48	2	5.81	5.67	2.48	50	0	5.82	5.67	2.64	50	0
$(0.2, 0.5)$	5.78	5.66	2.19	45	5	5.81	5.66	2.63	50	0	5.82	5.66	2.70	50	0
$(0.2, 0.6)$	5.78	5.66	2.05	46	4	5.81	5.66	2.47	49	1	5.82	5.66	2.67	50	0
$(0.3, 0.4)$	5.60	5.44	2.95	50	0	5.64	5.44	3.54	50	0	5.66	5.44	3.89	50	0
$(0.3, 0.5)$	5.64	5.46	3.17	50	0	5.66	5.46	3.60	50	0	5.67	5.46	3.84	50	0
$(0.3, 0.6)$	5.60	5.43	3.09	47	3	5.63	5.43	3.57	50	0	5.65	5.43	3.90	50	0
$(0.4, 0.4)$	5.51	5.25	4.62	50	0	5.53	5.26	4.98	50	0	5.54	5.26	5.19	50	0
$(0.4, 0.5)$	5.52	5.26	4.63	50	0	5.54	5.26	4.99	50	0	5.55	5.26	5.21	50	0
$(0.4, 0.6)$	5.50	5.25	4.63	50	0	5.53	5.24	5.09	50	0	5.54	5.24	5.29	50	0
$(0.5, 0.4)$	5.41	5.10	5.69	50	0	5.44	5.10	6.20	50	0	5.45	5.10	6.44	50	0
$(0.5, 0.5)$	5.41	5.10	5.75	50	0	5.43	5.10	6.22	50	0	5.45	5.09	6.48	50	0
$(0.5, 0.6)$	5.40	5.08	5.95	50	0	5.43	5.08	6.49	50	0	5.45	5.08	6.85	50	0
Average			3.90	48.83	1.17			4.35	49.92	0.08			4.59	50	0

4.7.2 Computational Benefits of the Compact Formulation

In this section, we check the computational benefits of using the Compact LP in conjunction with Theorem 4.6 to get an optimal solution to the Choice-Based LP, rather than solving the Choice-Based LP directly by using column generation.

Experimental Setup:

We generate multiple instances of the network revenue management problem using the following approach. The set of products is $N = \{1, \ldots, n\}$ with $n = 100$, and the set of resources is $M = \{1, \ldots, m\}$, where m is a parameter that we vary. In the multinomial logit model, for each product i, we generate η_i from the uniform distribution over $[0, 1]$ and set the preference weight of product i as $v_i = \eta_i \left(\frac{1-P_0}{P_0}\right) / \sum_{j \in N} \eta_j$, where P_0 is another parameter that we vary. In this case, if we offer all products, then the customer segment with the multinomial logit model leaves without a purchase with probability $\frac{1}{1+\sum_{i \in N} v_i} = \frac{1}{1+(1-P_0)/P_0} = P_0$. In the independent demand model, we generate γ_i from the uniform distribution over $[0, 1]$ and set the probability of demand for product i as $\theta_i = \gamma_i / \sum_{j \in N} \gamma_j$. In this case, the purchase probability of product i within assortment S is $\phi_i(S) = \beta \, \frac{v_i}{1+\sum_{j \in S} v_j} + (1-\beta)\, \theta_i$, where β is one more parameter that we vary.

We have $T = 100$ time periods. We sample the revenue r_i of each product i from the uniform distribution over $[100, 500]$. For each product i, we randomly choose a resource q_i and set $a_{q_i,i} = 1$. For the other resources, we set $a_{qi} = 1$ with probability $1/5$ and $a_{qi} = 0$ with probability $4/5$ for all $q \in M \setminus \{q_i\}$. Thus, the expected number of resources used by a product is $1 + (m-1)/5$. To come up with the capacities of the resources, noting that $\phi_i(S)$ is the choice probability of product i within assortment S in the previous paragraph, we let S^* be an optimal solution to the problem $\max_{S \subseteq N} \sum_{i \in S} r_i \, \phi_i(S)$, which is the assortment that maximizes the expected revenue under infinite resource capacities. If we offer the assortment S^* over the entire selling horizon, then the total expected capacity consumption of resource q is $T \sum_{i \in S^*} a_{qi} \, \phi_i(S^*)$. We set the capacity of resource q as $c_q = \kappa \, T \sum_{i \in S^*} a_{qi} \, \phi_i(S^*)$, where κ is a last parameter that we vary.

Computational Results:

Varying $(m, P_0, \beta, \kappa) \in \{25, 50\} \times \{0.1, 0.2\} \times \{0.25, 0.75\} \times \{0.6, 0.8\}$, we obtain 16 parameter combinations. For each parameter combination, we generate a problem instance by using the approach in the previous two paragraphs. We obtain an optimal solution to the Choice-Based LP for each problem instance by using two methods. First, we solve the Choice-Based LP directly by using column generation. We refer to this method as COG, which stands for column generation. Second, we solve the Compact LP and build on Theorem 4.6 to use an optimal solution of this LP to recover an optimal solution of the Choice-Based LP. We refer to this method as CLP, which stands for compact LP. We show our results in Table 4.4. The first column gives the parameter combination. The second column gives the running time for COG to obtain an optimal solution to the Choice-Based LP through column generation. The third column gives the running time for CLP to solve the Compact LP and use an optimal solution to this LP to recover an optimal solution to the Choice-Based LP. We use Gurobi 9.0 as our LP solver. The fourth column gives the ratio of the running times in the second and third columns. Column generation may get near-optimal solutions quickly but may take a while to close the remaining portion of the optimality gap. To check for this possibility, the fifth column gives the running time for COG to solve the Choice-Based LP with a 1% optimality gap. The sixth column gives the ratio of the running times in the third and fifth columns. Our results indicate that CLP can improve the running times for COG by up to a factor of 29.23. The average improvement in the running times is a factor of 15.88. If we allow COG to terminate with a 1% optimality gap, but run CLP until it gets to the optimal solution, then CLP can still improve the running times for COG by up to a factor of 9.05. The improvements in the running times become more pronounced when m is larger. In our test problems, most of the running time for COG is spent on solving the Compact LP. It takes less than one-tenths of a second to recover an optimal solution to the Choice-Based LP by using an optimal solution to the Compact LP through Theorem 4.6.

Table 4.4: Running times for solving the Choice-Based LP through two methods

Param. (m, P_0, β, κ)	COG Secs.	CLP Secs.	Secs. Ratio	1%Gp. COG Secs.	1%Gp. Secs. Ratio
$(25, 0.1, 0.25, 0.6)$	62.51	4.24	14.74	22.82	5.38
$(25, 0.1, 0.25, 0.8)$	58.34	5.09	11.46	21.06	4.14
$(25, 0.1, 0.75, 0.6)$	70.50	4.79	14.72	37.98	7.93
$(25, 0.1, 0.75, 0.8)$	72.07	7.72	9.34	31.94	4.14
$(25, 0.2, 0.25, 0.6)$	55.21	4.11	13.43	23.56	5.73
$(25, 0.2, 0.25, 0.8)$	52.23	5.37	9.73	20.03	3.73
$(25, 0.2, 0.75, 0.6)$	86.15	5.02	17.16	36.20	7.21
$(25, 0.2, 0.75, 0.8)$	74.90	9.80	7.64	31.43	3.21
Average			12.28		5.18
$(50, 0.1, 0.25, 0.6)$	101.38	6.02	16.84	29.63	4.92
$(50, 0.1, 0.25, 0.8)$	93.84	4.11	22.83	27.99	6.81
$(50, 0.1, 0.75, 0.6)$	128.20	6.95	18.45	46.15	6.64
$(50, 0.1, 0.75, 0.8)$	143.04	8.78	16.29	55.12	6.28
$(50, 0.2, 0.25, 0.6)$	139.19	5.15	27.03	29.62	5.75
$(50, 0.2, 0.25, 0.8)$	149.65	5.12	29.23	46.32	9.05
$(50, 0.2, 0.75, 0.6)$	145.13	10.32	14.06	55.17	5.35
$(50, 0.2, 0.75, 0.8)$	122.47	11.07	11.06	38.22	3.45
Average			19.47		6.03

We also compare the performance of COG and CLP for larger test problems with $n = 500$ products and $m = 100$ resources. For such test problems, COG does not reach an optimal solution within one hour of running time. We give our results in Table 4.5. The first column shows the problem parameters. The interpretation of the problem parameters P_0, β, and κ is the same as the one presented earlier in this section. The second column shows the optimality gap for COG after one hour of running time. The third column shows the running time for CLP to get the optimal solution. Over all the test problems, the average optimality gap of the solutions obtained by COG after one hour of running time is 8.56%. There are test problems for which COG terminates with more than a 14% optimality gap. The average running time for CLP to obtain an optimal solution is about 23 minutes, the longest running time not exceeding 41 minutes.

Table 4.5: Optimality gaps and running times for the two methods for solving the Choice-Based LP

Param. (P_0, β, κ)	COG % Opt. Gap.	CLP Secs.	Param. (P_0, β, κ)	COG % Opt. Gap.	CLP Secs.
$(0.1, 0.25, 0.6)$	4.93	676.13	$(0.2, 0.25, 0.6)$	7.17	408.59
$(0.1, 0.25, 0.8)$	5.27	2139.15	$(0.2, 0.25, 0.8)$	4.71	1059.42
$(0.1, 0.75, 0.6)$	14.27	909.10	$(0.2, 0.75, 0.6)$	14.10	950.74
$(0.1, 0.75, 0.8)$	9.51	2420.80	$(0.2, 0.75, 0.8)$	8.53	2458.98
Average	8.49	1536.29	Average	8.63	1219.43

4.8 Concluding Remarks

We studied the single-shot unconstrained and cardinality-constrained assortment optimization and assortment-based network revenue management problems under a mixture of multinomial logit and independent demand models. Our mixture choice model is a natural way to simultaneously improve the flexibility of both the multinomial logit and independent demand models to capture the choice process of the customers, while ensuring that the corresponding assortment optimization problems remain tractable.

There are several avenues for further research. Mixing the multinomial logit model with the independent demand model results in efficiently solvable assortment optimization problems, but our results closely exploit the structure of the multinomial logit model. One can mix the independent demand model with other choice models and try to tackle the corresponding assortment optimization problems. Moreover, we focus on solving assortment optimization problems, but our computational experiments indicate that our mixture choice model can improve the modeling flexibility of the pure multinomial logit model in terms of predicting the purchases of the customers. It would be useful to test the prediction ability of our mixture choice model on data generated by real-world applications. Lastly, one can work on enriching our mixture choice model by incorporating an incremental process for viewing the products in batches or by allowing customers with different consideration sets.

Appendices

APPENDIX A

APPENDIX FOR CHAPTER 2

A.1 Properties of the Spiked-MNL Model

There are several differences between the properties of the MNL and the spiked-MNL models. First we introduce some additional notations. For any product $j \in \mathcal{J}$, let $g(j)$ denote the itinerary that product j is associated with, and let $J(j) := \{j' \in \mathcal{J}^{g(j)} : r_{j'} > r_j\}$ denote the set of products associated with the same itinerary as product j and that have higher fares than product j. Let $\bar{J}(j) := J(j) \cup \{j\}$, and let $\underline{J}(j) := \{j' \in \mathcal{J}^{g(j)} : r_{j'} < r_j\}$ denote the set of products associated with the same itinerary as product j and that have lower fares than product j.

A.1.1 Regularity

The regularity of a choice model states that the probability of choosing any alternative, including the null alternative, from an assortment does not increase if the assortment is enlarged [101]. More formally, the definition of a regular choice model is as follows.

Definition A.1. *A choice model P is* regular *if for any two assortments S_1 and S_2 satisfying $S_1 \subset S_2 \subset \mathcal{J}$ and any alternative $j \in S_1 \cup \{0\}$, it holds that $P_{j:S_1} \geq P_{j:S_2}$.*

Regularity is a property commonly held by choice models in the assortment optimization literature [102, 103]. Although the MNL choice model is regular, the spiked-MNL choice model is not necessarily regular. Consider the following example:

Example A.2. *A seller sells three products H, M, and L with revenues $r_H > r_M > r_L$. Let the preference weights of these products be $v_H = v_M = w_L = 1$ and $w_M = 8$ (we don't need to specify v_L or w_H in this example), and let the null preference weight be $v_0 = 1$.*

Then $P_{H:\{H,M\}} = v_H/(v_H+w_M+v_0) = 1/10$ *and* $P_{H:\{H,M,L\}} = v_H/(v_H+v_M+w_L+v_0) = 1/4$*, which violates the regularity property.*

In order to check whether the spiked-MNL model is regular, or to enforce regularity when estimating a spiked-MNL model, we have the following necessary and sufficient condition, whose proof is in Appendix A.2 The complexity of checking the regularity of a spiked-MNL model is $O(\sum_{g\in\mathcal{G}} n(g)^2)$.

Proposition A.3. *The spiked-MNL model is regular if and only if for any two products* j *and* j' *for the same itinerary, with* j' *more expensive than* j*, i.e., for any* $j' \in J(j)$*, it holds that*

$$w_j + v_{j'} \;\geq\; w_{j'}.$$

A.1.2 Submodularity

Given a choice model, let the *demand function* $d : 2^{\mathcal{J}} \mapsto \mathbb{R}$ of the choice model be given by $d(S) := \sum_{j\in S} P_{j:S}$ for any assortment $S \subset \mathcal{J}$. Another property of many choice models is the submodularity of their demand functions, which means that the marginal increment in total choice probability decreases as the assortment enlarges [103]. More formally, the definition of a submodular demand function is as follows.

Definition A.4. *The demand function* d *of a choice model is* submodular *if*

$$d(S_2 \cup \{k\}) - d(S_2) \;\leq\; d(S_1 \cup \{k\}) - d(S_1), \quad \forall\, S_1 \subset S_2 \subset \mathcal{J}, k \in \mathcal{J} \setminus S_2. \tag{A.1}$$

The demand function of the MNL choice model is submodular, but the demand function of the spiked-MNL choice model is not necessarily submodular. Consider the following example:

Example A.5. *A seller sells three products H, M, and L with revenues* $r_H > r_M > r_L$*. Let the preference weights of the products be* $v_H = 1$*,* $w_H = 3$*, and* $v_M = w_M = w_L = 2$

(we don't need to specify v_L); and let the null preference weight be $v_0 = 1$. Consider set $S_1 = \{H\}$, set $S_2 = \{H, L\}$, and product $k = M$. Then $d(S_2 \cup \{k\}) - d(S_2) = d(\{H, M, L\}) - d(\{H, L\}) = 5/6 - 3/4 = 1/12$, and $d(S_1 \cup \{k\}) - d(S_1) = d(\{H, M\}) - d(\{H\}) = 3/4 - 3/4 = 0$. Therefore, the demand function is not submodular.

Any random utility model has a submodular demand function and is equivalent to a certain stochastic preference model [103]. Our example shows that a spiked-MNL model is not representable by a random utility model or stochastic preference model in general.

A.1.3 Revenue Gap between the MNL and the Spiked-MNL Choice Models

Next we show that using an MNL model instead of a spiked-MNL model in the presence of the cheapest-fare spike effect can lead to arbitrarily bad relative revenue performance.

Setting. A seller can offer two products H and L with no capacity limits. The products H and L sell at prices r_H and $r_L > 0$ respectively, with $0 < r_L/r_H \leq \varepsilon$. We consider a sequence of selling seasons, indexed by $k = 1, 2, 3, \ldots$. In each season k, the seller offers one assortment $A^{(k)} \subset \{H, L\}$. For reasons that will be explored in Section 2.5, it suffices to consider either $\{H\}$ or $\{H, L\}$ for $A^{(k)}$. Customers make choices according to a spiked-MNL model with parameters $w_H = w_L > 0$, $v_H = 0$ and $v_0 = 1$. Note that $v_H = 0$ corresponds to the so-called 100% buydown effect, where customers buy only product L when both H and L are offered. It follows that $r_H w_H/(v_0 + w_H) > (r_H v_H + r_L w_L)/(v_0 + v_H + w_L)$, and thus it is optimal to offer assortment $\{H\}$. In addition, we assume that

$$w_H < \eta := \frac{r_L}{r_H - r_L}.$$

As process primitives, consider the following 4 independent, i.i.d. sequences of random variables: $N_1^{(k)}$ with mean $\lambda w_H/(v_0 + w_H)$, representing the number of customers who would choose H in season k if assortment $\{H\}$ is offered; $N_2^{(k)}$ with mean $\lambda v_0/(v_0 + w_H)$, representing the number of customers who would choose 0 in season k if assortment $\{H\}$ is offered; $N_3^{(k)}$ with mean $\lambda w_L/(v_0 + v_H + w_L)$, representing the number of customers who would choose L in season k if assortment $\{H, L\}$ is offered; $N_4^{(k)}$ with mean $\lambda v_0/(v_0 + v_H + w_L)$, representing the number of customers who would choose 0 in season k if assortment $\{H, L\}$ is offered.

Dynamics. After each season, the revenue manager calibrates an MNL model using maximum likelihood estimation (MLE) with all historical sales data (including no-purchase customers), and decides which assortment to offer in the next season based on the estimated MNL model. The MNL model is specified with preference weights $\tilde{v}_H$, $\tilde{v}_L$, and $v_0 = 1$. For each season k, let $n_H^{(k)}$ denote the sales of product H and let $n_0^{(k)}$ denote the number of customers who choose not to purchase. One can show the following:

- Regardless of the assortments offered, the MLE estimated preference weight of product H is given by $\tilde{v}_H^{(k)} = \sum_{k'=1}^{k} n_H^{(k')} / \sum_{k'=1}^{k} n_0^{(k')}$. (To deal with the possibility that the denominator may be zero for the first few seasons, assume that the denominator is set to 1 if $\sum_{k'=1}^{k} n_0^{(k')} = 0$.)

- According to the estimated MNL model, it is optimal to offer assortment $\{H\}$ if $\tilde{v}_H^{(k)} \geq \eta$; otherwise, it is optimal to offer assortment $\{H, L\}$.

Next, we show that under the MNL model, the offered assortment converges to $\{H, L\}$ w.p.1. The long-run revenue loss ratio under this control is greater $1 - \varepsilon$. We consider two cases.

Case 1: The revenue manager offers assortment $A^{(1)} = \{H, L\}$ in the first season. Then, due to 100% buydown, it follows that $n_H^{(1)} = 0$. Thus, the estimated parameter $\tilde{v}_H^{(1)} = n_H^{(1)}/n_0^{(1)} = 0 < \eta$. It follows by induction that the revenue manager will offer assortment $\{H, L\}$ and $\tilde{v}_H^{(k)} = 0$ for all k.

Case 2: The revenue manager offers assortment $A^{(1)} = \{H\}$ in the first season. Note that, if in any season k it holds that $\tilde{v}_H^{(k)} < \eta$, then the revenue manager will offer assortment $A^{(k+1)} = \{H, L\}$ in season $k+1$. Subsequently, it follows that $n_H^{(k+1)} = 0$ and thus $\tilde{v}_H^{(k+1)} = \sum_{k'=1}^{k+1} n_H^{(k')} / \sum_{k'=1}^{k+1} n_0^{(k')} = \left(\sum_{k'=1}^{k} n_H^{(k')} + 0\right) / \left(\sum_{k'=1}^{k} n_0^{(k')} + N_4^{(k+1)}\right) \leq$

$\sum_{k'=1}^{k} n_H^{(k')} / \sum_{k'=1}^{k} n_0^{(k')} = \tilde{v}_H^{(k)} < \eta$, and it follows by induction that $A^{(k')} = \{H, L\}$ for all $k' > k$. Thus, either there is a K such that $A^{(k)} = \{H, L\}$ for all $k > K$, or $\tilde{v}_H^{(k)} \geq \eta$ and $A^{(k)} = \{H\}$ for all k. Next we show that the event that $\tilde{v}_H^{(k)} \geq \eta$ and $A^{(k)} = \{H\}$ for all k has probability 0. Note that if $A^{(k)} = \{H\}$ for all k, then $n_H^{(k)} = N_1^{(k)}$ and $n_0^{(k)} = N_2^{(k)}$ for all k. By the Strong Law of Large Numbers, w.p.1, $\sum_{k'=1}^{k} N_1^{(k')}/k \to \lambda w_H/(v_0 + w_H)$ and $\sum_{k'=1}^{k} N_2^{(k)}/k \to \lambda v_0/(v_0 + w_H)$ as $k \to \infty$. Thus, if $A^{(k)} = \{H\}$ for all k, then, except for a subset B with probability 0, it holds that $\sum_{k'=1}^{k} n_H^{(k')}/k = \sum_{k'=1}^{k} N_1^{(k')}/k \to \lambda w_H/(v_0 + w_H)$ and $\sum_{k'=1}^{k} n_0^{(k')}/k = \sum_{k'=1}^{k} N_2^{(k)}/k \to \lambda v_0/(v_0 + w_H)$ as $k \to \infty$, and hence

$$\tilde{v}_H^{(k)} \;=\; \frac{\sum_{k'=1}^{k} n_H^{(k')}}{\sum_{k'=1}^{k} n_0^{(k')}} \;\to\; \frac{w_H}{v_0} \;<\; \eta.$$

Therefore, the event that $\tilde{v}_H^{(k)} \geq \eta$ and $A^{(k)} = \{H\}$ for all k is contained in the subset B and thus has probability 0.

Loss Ratio. In both cases above, the long-run loss ratio is

$$\text{Loss} \;=\; \frac{R_{\text{SMNL}} - R_{\text{MNL}}}{R_{\text{SMNL}}} \;=\; \frac{R(\{H\}) - R(\{H, L\})}{R(\{H\})} \;=\; 1 - \frac{r_L}{r_H} \;\geq\; 1 - \varepsilon.$$

A.2 Proof of Proposition A.3

For any alternative $j \in \mathcal{J} \cup \{0\}$ and assortment $A \subset \mathcal{J}$, let

$$\tilde{v}(j, A) \;:=\; \begin{cases} w_j \mathbf{I}(j, A) + v_j(1 - \mathbf{I}(j, A)) & \text{if } j \in A \\ v_0 & \text{if } j = 0 \\ 0 & \text{otherwise} \end{cases} \tag{A.2}$$

denote the preference weight of j when A is offered, and let $\tilde{v}(A) := \sum_{j \in A} \tilde{v}(j, A)$ denote the total preference weight of A.

Proof of Proposition A.3. (*Sufficiency.*) Suppose that, for any two products j and j' for the same itinerary, with $j' \in J(j)$, it holds that

$$w_j + v_{j'} \;\geq\; w_{j'}.$$

We show that the spiked-MNL model is regular, that is, we show that for any two sets S' and S satisfying $S' \subset S \subset \mathcal{J}$, it holds that $P_{j:S'} \geq P_{j:S}$ for all $j \in S' \cup \{0\}$. Note that there is a nested sequence of sets $S' = S_0 \subset S_1 \subset S_2 \subset \cdots \subset S_{|S \setminus S'|-1} \subset S_{|S \setminus S'|} = S$ such that $|S_i \setminus S_{i-1}| = 1$ for all $i = 1, \ldots, |S \setminus S'|$. Therefore, we consider the case in which $|S \setminus S'| = 1$, and the regularity follows for general S' and S by induction. Let $S \setminus S' = \{k\}$.

Case 1: $k \in J(j)$ for some $j \in S'$. Then the cheapest available fare class on each itinerary remains the same when k is added to assortment S'. Then regularity holds as for the MNL model.

Case 2.1: $k \notin J(j)$ for any $j \in S'$, and $J(k) \cap S' = \varnothing$. This is similar to Case 1. Alternative k is the cheapest (and only) fare class for its itinerary in S. The cheapest available fare class on each other itinerary remains the same when k is added to assortment S'. Then regularity holds as for the MNL model.

Case 2.2: $k \notin J(j)$ for any $j \in S'$, and $J(k) \cap S' \neq \varnothing$. Alternative k is the cheapest fare class for its itinerary in S. Let $l \in J(k) \cap S'$ denote the cheapest fare class for itinerary $g(k)$ in assortment S'. Recall that $w_k + v_l \geq w_l$ by assumption, and thus $\tilde{v}(S) - \tilde{v}(S') = w_k + v_l - w_l \geq 0$, i.e., $\tilde{v}(S) \geq \tilde{v}(S') \geq 0$. Next we consider three cases:

If $j = l$, then $\tilde{v}(j, S') = w_l \geq v_l = \tilde{v}(j, S)$.

If $j \in S' \setminus \{l\}$, then $\tilde{v}(j, S') = v_j(1 - \mathbf{I}(j, S')) + w_j \mathbf{I}(j, S') = v_j(1 - \mathbf{I}(j, S)) + w_j \mathbf{I}(j, S) = \tilde{v}(j, S)$.

If $j = 0$, then $\tilde{v}(j, S') = \tilde{v}(j, S) = v_0$.

Therefore, for any $j \in S' \cup \{0\}$, it holds that $\tilde{v}(j, S') \geq \tilde{v}(j, S)$. It follows that

$$P_{j:S'} \;=\; \frac{\tilde{v}(j, S')}{\tilde{v}(S') + v_0} \;\geq\; \frac{\tilde{v}(j, S)}{\tilde{v}(S) + v_0} \;=\; P_{j:S} \quad \forall\, j \in S' \cup \{0\}.$$

(*Necessity.*) Suppose that there are two products j and j' for the same itinerary, with $j' \in J(j)$, such that

$$w_j + v_{j'} \; < \; w_{j'}.$$

We show that in such a case the spiked-MNL model is not regular. Let $S' = \{j'\}$ and $S = \{j, j'\}$. Then

$$P_{0:S'} \; = \; \frac{v_0}{v_0 + w_{j'}} \; < \; \frac{v_0}{v_0 + w_j + v_{j'}} \; = \; P_{0:S},$$

and thus the spiked-MNL model is not regular. $\square$

A.3 Proof of Lemma 2.2 and Theorem 2.4

Proposition A.6 ([17])**.** *An assortment $S \subset \mathcal{J}$ is efficient if and only if for some $\boldsymbol{\pi} \in \mathbb{R}^m_+$, set S is an optimal solution of the problem* $\max_{A \subset \mathcal{J}} \; \{R(A) - \boldsymbol{\pi}^\top \boldsymbol{Q}(A)\}$.

Proof of Lemma 2.2. If a set S is efficient, then by Proposition A.6, there exists $\boldsymbol{\pi} \in \mathbb{R}^m_+$ such that S is an optimal solution of $\max_{A \subset \mathcal{J}} \{R(A) - \boldsymbol{\pi}^\top \boldsymbol{Q}(A)\}$. Note that

$$R(A) - \boldsymbol{\pi}^\top \boldsymbol{Q}(A) \; = \; \sum_{j \in A} \left(r_j P_{j:A} - \boldsymbol{\pi}^\top \boldsymbol{a}^j P_{j:A}\right) \; = \; \sum_{j \in A} \gamma_j P_{j:A},$$

where $\gamma_j := r_j - \boldsymbol{\pi}^\top \boldsymbol{a}^j$. Note that if $j \in \mathcal{J}$ and $j' \in \underline{J}(j)$, then j and j' are associated with the same itinerary, and thus $\boldsymbol{a}^j = \boldsymbol{a}^{j'}$, hence $\boldsymbol{\pi}^\top \boldsymbol{a}^j = \boldsymbol{\pi}^\top \boldsymbol{a}^{j'}$, and $\gamma_j > \gamma_{j'}$ since $r_j > r_{j'}$. Therefore, optimization problem $\max_{A \subset \mathcal{J}} \{R(A) - \boldsymbol{\pi}^\top \boldsymbol{Q}(A)\}$ is reduced to the optimization problem in (2.2). $\square$

Next we repeat the definition of efficient sets with specific reference to the set of products considered.

Definition A.7 (Relatively Efficient Sets)**.** *An assortment $S \subset \mathcal{J}$ is said to be* inefficient relative to $\mathcal{J}$ *if a mixture of other assortments in $\mathcal{J}$ has strictly higher expected revenue* *with the same or lower expected resource consumption. That is, there exists a set of weights* $\{\mu(A) : A \subset \mathcal{J}\}$ *satisfying* $\sum_{A \subset \mathcal{J}} \mu(A) = 1$ *and* $\mu(A) \geq 0$ *for all* $A \subset \mathcal{J}$ *such that*

$$
\begin{aligned}
R(S) &:= \sum_{h \in \mathcal{H}} \frac{\lambda_h}{\sum_{h' \in \mathcal{H}} \lambda_{h'}} \sum_{j \in S \cap \mathcal{J}(h)} r_j P^h_{j:S \cap \mathcal{J}(h)} \\
&< \sum_{A \subset \mathcal{J}} \mu(A) R(A) \ := \ \sum_{A \subset \mathcal{J}} \mu(A) \sum_{h \in \mathcal{H}} \frac{\lambda_h}{\sum_{h' \in \mathcal{H}} \lambda_{h'}} \sum_{j \in A \cap \mathcal{J}(h)} r_j P^h_{j:A \cap \mathcal{J}(h)} \\
Q_f(S) &:= \sum_{h \in \mathcal{H}} \frac{\lambda_h}{\sum_{h' \in \mathcal{H}} \lambda_{h'}} \sum_{j \in S \cap \mathcal{J}(h)} a^j_f P^h_{j:S \cap \mathcal{J}(h)} \\
&\geq \sum_{A \subset \mathcal{J}} \mu(A) Q_f(A) \ := \ \sum_{A \subset \mathcal{J}} \mu(A) \sum_{h \in \mathcal{H}} \frac{\lambda_h}{\sum_{h' \in \mathcal{H}} \lambda_{h'}} \sum_{j \in A \cap \mathcal{J}(h)} a^j_f P^h_{j:A \cap \mathcal{J}(h)} \quad \forall f.
\end{aligned}
$$

An assortment in $\mathcal{J}$ *that is not inefficient relative to* $\mathcal{J}$ *is said to be* efficient relative to $\mathcal{J}$.

Next we show that a necessary (but not sufficient) condition for an assortment S to be efficient relative to $\mathcal{J}$ is that $S \cap \mathcal{J}(h)$ is efficient relative to $\mathcal{J}(h)$ for each h.

Lemma A.8. *If an assortment* S *is efficient relative to* $\mathcal{J}$*, then* $S \cap \mathcal{J}(h)$ *is efficient relative to* $\mathcal{J}(h)$ *for each* $h \in \mathcal{H}$.

Proof. Suppose that there is an $h' \in \mathcal{H}$ such that $S \cap \mathcal{J}(h')$ is inefficient relative to $\mathcal{J}(h')$, that is, there is a set of weights $\{\mu'(A') : A' \subset \mathcal{J}(h')\}$ satisfying $\sum_{A' \subset \mathcal{J}(h')} \mu'(A') = 1$ and $\mu'(A') \geq 0$ for all $A' \subset \mathcal{J}(h')$ such that

$$
\begin{aligned}
\sum_{j \in S \cap \mathcal{J}(h')} r_j P^{h'}_{j:S \cap \mathcal{J}(h')} &< \sum_{A' \subset \mathcal{J}(h')} \mu'(A') \sum_{j \in A'} r_j P^{h'}_{j:A'} \\
\sum_{j \in S \cap \mathcal{J}(h')} a^j_f P^{h'}_{j:S \cap \mathcal{J}(h')} &\geq \sum_{A' \subset \mathcal{J}(h')} \mu'(A') \sum_{j \in A'} a^j_f P^{h'}_{j:A'} \quad \forall f.
\end{aligned}
$$

Then we show that S is inefficient relative to $\mathcal{J}$. Consider the set of weights $\{\mu(A) : A \subset \mathcal{J}\}$ constructed as follows. For each $A' \subset \mathcal{J}(h')$, let $A := A' \cup (S \setminus \mathcal{J}(h'))$ and let $\mu(A) = \mu'(A')$. For all other $B \subset \mathcal{J}$, let $\mu(B) = 0$. Then $\sum_{A \subset \mathcal{J}} \mu(A) = \sum_{A' \subset \mathcal{J}(h')} \mu'(A') = 1$

and $\mu(A) \geq 0$ for all $A \subset \mathcal{J}$. Also,

$$
\begin{aligned}
R(S) :=& \frac{\lambda_{h'}}{\sum_{h'' \in \mathcal{H}} \lambda_{h''}} \sum_{j \in S \cap \mathcal{J}(h')} r_j P^{h'}_{j:S \cap \mathcal{J}(h')} \\
&\qquad + \sum_{\{h \in \mathcal{H} \,:\, h \neq h'\}} \frac{\lambda_h}{\sum_{h'' \in \mathcal{H}} \lambda_{h''}} \sum_{j \in S \cap \mathcal{J}(h)} r_j P^{h}_{j:S \cap \mathcal{J}(h)} \\
<& \frac{\lambda_{h'}}{\sum_{h'' \in \mathcal{H}} \lambda_{h''}} \sum_{A' \subset \mathcal{J}(h')} \mu'(A') \sum_{j \in A'} r_j P^{h'}_{j:A'} \\
&\qquad + \sum_{\{h \in \mathcal{H} \,:\, h \neq h'\}} \frac{\lambda_h}{\sum_{h'' \in \mathcal{H}} \lambda_{h''}} \sum_{j \in S \cap \mathcal{J}(h)} r_j P^{h}_{j:S \cap \mathcal{J}(h)} \\
=& \frac{\lambda_{h'}}{\sum_{h'' \in \mathcal{H}} \lambda_{h''}} \sum_{A \subset \mathcal{J}} \mu(A) \sum_{j \in A \cap \mathcal{J}(h')} r_j P^{h'}_{j:A \cap \mathcal{J}(h')} \\
&\qquad + \sum_{\{h \in \mathcal{H} \,:\, h \neq h'\}} \frac{\lambda_h}{\sum_{h'' \in \mathcal{H}} \lambda_{h''}} \sum_{A \subset \mathcal{J}} \mu(A) \sum_{j \in A \cap \mathcal{J}(h)} r_j P^{h}_{j:A \cap \mathcal{J}(h)} \\
=& \sum_{A \subset \mathcal{J}} \mu(A) \sum_{h \in \mathcal{H}} \frac{\lambda_h}{\sum_{h'' \in \mathcal{H}} \lambda_{h''}} \sum_{j \in A \cap \mathcal{J}(h)} r_j P^{h}_{j:A \cap \mathcal{J}(h)}
\end{aligned}
$$

and

$$
\begin{aligned}
Q_f(S) :=& \frac{\lambda_{h'}}{\sum_{h'' \in \mathcal{H}} \lambda_{h''}} \sum_{j \in S \cap \mathcal{J}(h')} a^j_f P^{h'}_{j:S \cap \mathcal{J}(h')} \\
&\qquad + \sum_{\{h \in \mathcal{H} \,:\, h \neq h'\}} \frac{\lambda_h}{\sum_{h'' \in \mathcal{H}} \lambda_{h''}} \sum_{j \in S \cap \mathcal{J}(h)} a^j_f P^{h}_{j:S \cap \mathcal{J}(h)} \\
\geq& \frac{\lambda_{h'}}{\sum_{h'' \in \mathcal{H}} \lambda_{h''}} \sum_{A' \subset \mathcal{J}(h')} \mu'(A') \sum_{j \in A'} a^j_f P^{h'}_{j:A'} \\
&\qquad + \sum_{\{h \in \mathcal{H} \,:\, h \neq h'\}} \frac{\lambda_h}{\sum_{h'' \in \mathcal{H}} \lambda_{h''}} \sum_{j \in S \cap \mathcal{J}(h)} a^j_f P^{h}_{j:S \cap \mathcal{J}(h)} \\
=& \frac{\lambda_{h'}}{\sum_{h'' \in \mathcal{H}} \lambda_{h''}} \sum_{A \subset \mathcal{J}} \mu(A) \sum_{j \in A \cap \mathcal{J}(h')} a^j_f P^{h'}_{j:A \cap \mathcal{J}(h')} \\
&\qquad + \sum_{\{h \in \mathcal{H} \,:\, h \neq h'\}} \frac{\lambda_h}{\sum_{h'' \in \mathcal{H}} \lambda_{h''}} \sum_{A \subset \mathcal{J}} \mu(A) \sum_{j \in A \cap \mathcal{J}(h)} a^j_f P^{h}_{j:A \cap \mathcal{J}(h)} \\
=& \sum_{A \subset \mathcal{J}} \mu(A) \sum_{h \in \mathcal{H}} \frac{\lambda_h}{\sum_{h'' \in \mathcal{H}} \lambda_{h''}} \sum_{j \in A \cap \mathcal{J}(h)} a^j_f P^{h}_{j:A \cap \mathcal{J}(h)} \quad \text{for all } f \in \mathcal{F}
\end{aligned}
$$

and thus S is inefficient relative to $\mathcal{J}$. $\square$

Lemma A.8 shows that a necessary condition for an assortment S to be efficient relative to $\mathcal{J}$ is that $S \cap \mathcal{J}(h)$ is efficient relative to $\mathcal{J}(h)$ for each h. Next we show that a necessary condition for $S \cap \mathcal{J}(h)$ to be efficient relative to $\mathcal{J}(h)$ is that $S \cap \mathcal{J}(h)$ is nested by revenue. First we streamline some notation. For a fixed assortment $S' \subset \mathcal{J}$ and a fixed market $h \in \mathcal{H}$, let $A := S' \cap \mathcal{J}(h)$. For any $S \subset \mathcal{J}(h)$ and for a fixed itinerary $g \in \mathcal{G}$, let $S^g := S \cap \mathcal{J}^g$, and let $S^{-g} := S \setminus S^g$. Also, let $v(S) := \sum_{j \in S} v_j$, and let

$$
\begin{aligned}
\gamma(S) &:= \sum_{j \in S} \gamma_j v_j & \tilde{\gamma}(S) &:= \sum_{j \in S} \gamma_j \tilde{v}(j, S) \\
\Gamma(S^g, S^{-g}) &:= \frac{\gamma(S^g) + \tilde{\gamma}(S^{-g})}{v(S^g) + \tilde{v}(S^{-g}) + v_0} & \tilde{\Gamma}(S) &:= \sum_{j \in S} \gamma_j P_{j:S} = \frac{\tilde{\gamma}(S)}{\tilde{v}(S) + v_0},
\end{aligned}
$$

where $\tilde{v}(j, S)$ follows the definition in (A.2) and $\tilde{v}(S) := \sum_{j \in S} \tilde{v}(j, S)$.

Lemma A.9. *Consider any assortment $S' \subset \mathcal{J}$, any market $h \in \mathcal{H}$, and any itinerary $g \in \mathcal{G}$. Let $n' := |A^g|$, and index the products in A^g such that $\{r_j, j \in A^g\} = \{r_{1,g}, \ldots, r_{n',g}\}$ and $r_{1,g} > \cdots > r_{i,g} > \cdots > r_{n',g}$. Let $A_0^g := \varnothing$, and for $i = 1, \ldots, n'$, let $A_i^g := \{(1, g), (2, g), \ldots, (i, g)\} \subset A^g$. For any $i \in \{1, \ldots, n'\}$, it holds that $\tilde{\Gamma}(A_i^g \cup A^{-g}) < \Gamma(A_i^g, A^{-g})$ if and only if $\Gamma(A_i^g, A^{-g}) < \Gamma(A_{i-1}^g, A^{-g})$.*

Proof. It follows from $w_{(i,g)} \geq v_{(i,g)}$ that

$$
\begin{aligned}
& \tilde{\Gamma}(A_i^g \cup A^{-g}) < \Gamma(A_i^g, A^{-g}) \\
\Leftrightarrow \quad & \frac{\gamma_{(i,g)} w_{(i,g)} + \gamma(A_{i-1}^g) + \tilde{\gamma}(A^{-g})}{w_{(i,g)} + v(A_{i-1}^g) + \tilde{v}(A^{-g}) + v_0} < \frac{\gamma_{(i,g)} v_{(i,g)} + \gamma(A_{i-1}^g) + \tilde{\gamma}(A^{-g})}{v_{(i,g)} + v(A_{i-1}^g) + \tilde{v}(A^{-g}) + v_0} \\
\Leftrightarrow \quad & \gamma_{(i,g)} < \frac{\gamma(A_{i-1}^g) + \tilde{\gamma}(A^{-g})}{v(A_{i-1}^g) + \tilde{v}(A^{-g}) + v_0} \\
\Leftrightarrow \quad & \frac{\gamma_{(i,g)} v_{(i,g)} + \gamma(A_{i-1}^g) + \tilde{\gamma}(A^{-g})}{v_{(i,g)} + v(A_{i-1}^g) + \tilde{v}(A^{-g}) + v_0} < \frac{\gamma(A_{i-1}^g) + \tilde{\gamma}(A^{-g})}{v(A_{i-1}^g) + \tilde{v}(A^{-g}) + v_0} \\
\Leftrightarrow \quad & \Gamma(A_i^g, A^{-g}) < \Gamma(A_{i-1}^g, A^{-g}).
\end{aligned}
$$

$\square$

Lemma A.10. *For any assortment $S \subset \mathcal{J}$ and any market $h \in \mathcal{H}$, if $S \cap \mathcal{J}(h)$ is efficient relative to $\mathcal{J}(h)$, then $S \cap \mathcal{J}(h)$ is nested by revenue.*

Proof. By contradiction. Assume there is an assortment S' and a market h such that $S' \cap \mathcal{J}(h)$ is efficient relative to $\mathcal{J}(h)$ but $S' \cap \mathcal{J}(h)$ is not nested by revenue. Since $S' \cap \mathcal{J}(h)$ is efficient relative to $\mathcal{J}(h)$, it follows from Lemma 2.2 that there is a $\boldsymbol{\gamma}$ such that $\gamma_j > \gamma_{j'}$ for all $j \in \mathcal{J}(h)$ and all $j' \in \underline{J}(j)$, and

$$A \;:=\; S' \cap \mathcal{J}(h) \;\in\; \underset{S \subset \mathcal{J}(h)}{\arg\max} \left\{ \tilde{\Gamma}(S) \;:=\; \sum_{j \in S} \gamma_j P_{j:S} \right\}. \tag{A.3}$$

Since A is not nested by revenue, there is an itinerary $g \in \mathcal{G}$ and a product $j \in \mathcal{J}^g$ such that j is not offered in A, but A includes a product for the same itinerary g as j that has lower revenue than j, that is, $j \notin A$ and $A^g \cap \underline{J}(j) \neq \varnothing$.

We show that $\tilde{\Gamma}(A \cup \{j\}) > \tilde{\Gamma}(A)$, i.e.,

$$\begin{aligned} &\frac{\gamma(A^g \cap J(j)) + \gamma_j v_j + \tilde{\gamma}(A^g \cap \underline{J}(j)) + \tilde{\gamma}(A^{-g})}{v(A^g \cap J(j)) + v_j + \tilde{v}(A^g \cap \underline{J}(j)) + \tilde{v}(A^{-g}) + v_0} \\ &\qquad\qquad > \; \frac{\gamma(A^g \cap J(j)) + \tilde{\gamma}(A^g \cap \underline{J}(j)) + \tilde{\gamma}(A^{-g})}{v(A^g \cap J(j)) + \tilde{v}(A^g \cap \underline{J}(j)) + \tilde{v}(A^{-g}) + v_0}, \end{aligned}$$

which gives

$$\gamma_j \;>\; \frac{\gamma(A^g \cap J(j)) + \tilde{\gamma}(A^g \cap \underline{J}(j)) + \tilde{\gamma}(A^{-g})}{v(A^g \cap J(j)) + \tilde{v}(A^g \cap \underline{J}(j)) + \tilde{v}(A^{-g}) + v_0}.$$

Since $\gamma_j > \gamma_{j'}$ for all $j \in \mathcal{J}(h)$ and all $j' \in \underline{J}(j)$, it follows that $\gamma_j > \tilde{\gamma}(A^g \cap \underline{J}(j))/\tilde{v}(A^g \cap \underline{J}(j))$. Thus, we can show the above inequality by showing

$$\begin{aligned} &\frac{\tilde{\gamma}(A^g \cap \underline{J}(j))}{\tilde{v}(A^g \cap \underline{J}(j))} \;\geq\; \frac{\gamma(A^g \cap J(j)) + \tilde{\gamma}(A^g \cap \underline{J}(j)) + \tilde{\gamma}(A^{-g})}{v(A^g \cap J(j)) + \tilde{v}(A^g \cap \underline{J}(j)) + \tilde{v}(A^{-g}) + v_0} \\ \Leftrightarrow\;\; &\frac{\tilde{\gamma}(A^g \cap \underline{J}(j))}{\tilde{v}(A^g \cap \underline{J}(j))} \;\geq\; \frac{\gamma(A^g \cap J(j)) + \tilde{\gamma}(A^{-g})}{v(A^g \cap J(j)) + \tilde{v}(A^{-g}) + v_0}. \end{aligned}$$

Consider two cases:

Case 1: $\tilde{\Gamma}((A^g \cap J(j)) \cup A^{-g}) \ \geq \ \Gamma(A^g \cap J(j), A^{-g})$.

It follows from (A.3) that $\tilde{\Gamma}(A) \geq \tilde{\Gamma}((A^g \cap J(j)) \cup A^{-g})$. Thus $\tilde{\Gamma}(A) \geq \Gamma(A^g \cap J(j), A^{-g})$.

Case 2: $\tilde{\Gamma}((A^g \cap J(j)) \cup A^{-g}) \ < \ \Gamma(A^g \cap J(j), A^{-g})$.

In this case $A^g \cap J(j) \neq \varnothing$. Let $n' := |A^g \cap J(j)|$, and index the products in $A^g \cap J(j)$ such that $\{r_j, j \in A^g \cap J(j)\} = \{r_{1,g}, \ldots, r_{n',g}\}$ and $r_{1,g} > \cdots > r_{i,g} > \cdots > r_{n',g}$. Let $A_0^g := \varnothing$, and for $i = 1, \ldots, n'$, let $A_i^g := \{(1,g), (2,g), \ldots, (i,g)\} \subset A^g \cap J(j)$. Let $i^* := \max\left\{i \in \{0, 1, \ldots, n'-1\} \ : \ \tilde{\Gamma}(A_i^g \cup A^{-g}) \geq \Gamma(A_i^g, A^{-g})\right\}$. Repeatedly applying Lemma A.9, it follows that

$$\begin{aligned}\Gamma(A^g \cap J(j), A^{-g}) \ &= \ \Gamma(A_{n'}^g, A^{-g}) \ < \ \Gamma(A_{n'-1}^g, A^{-g}) \\ &< \ \cdots \ < \ \Gamma(A_{i^*}^g, A^{-g}) \ \leq \ \tilde{\Gamma}(A_{i^*}^g \cup A^{-g}) \ \leq \ \tilde{\Gamma}(A).\end{aligned}$$

Thus, in both cases, it holds that

$$\begin{aligned}&\tilde{\Gamma}(A) \ \geq \ \Gamma(A^g \cap J(j), A^{-g}) \\ \Leftrightarrow \ &\frac{\gamma(A^g \cap J(j)) + \tilde{\gamma}(A^g \cap \underline{J}(j)) + \tilde{\gamma}(A^{-g})}{v(A^g \cap J(j)) + \tilde{v}(A^g \cap \underline{J}(j)) + \tilde{v}(A^{-g}) + v_0} \ \geq \ \frac{\gamma(A^g \cap J(j)) + \tilde{\gamma}(A^{-g})}{v(A^g \cap J(j)) + \tilde{v}(A^{-g}) + v_0} \\ \Leftrightarrow \ &\frac{\tilde{\gamma}(A^g \cap \underline{J}(j))}{\tilde{v}(A^g \cap \underline{J}(j))} \ \geq \ \frac{\gamma(A^g \cap J(j)) + \tilde{\gamma}(A^{-g})}{v(A^g \cap J(j)) + \tilde{v}(A^{-g}) + v_0}.\end{aligned}$$

We have shown that $\tilde{\Gamma}(A \cup \{j\}) > \tilde{\Gamma}(A)$, which contradicts (A.3). □

Proof of Theorem 2.4. Lemma A.8 shows that if assortment $S \subset \mathcal{J}$ is efficient (relative to $\mathcal{J}$), then $S \cap \mathcal{J}(h)$ is efficient relative to $\mathcal{J}(h)$ for each h. Lemma A.10 shows that if $S \cap \mathcal{J}(h)$ is efficient relative to $\mathcal{J}(h)$, then $S \cap \mathcal{J}(h)$ is nested by revenue. Therefore, all efficient sets under the spiked-MNL choice model are nested by revenue. □

A.4 Proof of Theorem 2.7

A.4.1 From SBLP to CDLP

We provide a polynomial time algorithm (Algorithm 1) to convert a feasible solution of the SBLP (2.4), denoted $(\boldsymbol{x}, \boldsymbol{x}_0)$, into a feasible solution of the CDLP (2.3), denoted $\boldsymbol{\alpha}$, with the same objective value.

Algorithm 1 Converting a SBLP solution to a CDLP solution

Require: SBLP solution $(\boldsymbol{x}, \boldsymbol{x}_0)$

Set $\boldsymbol{\alpha} \leftarrow 0$, $k \leftarrow 1$
while there exists $j \in \mathcal{J}$ such that $x_j > 0$ **do**
  for all $h \in \mathcal{H}$ **do**
    if $x_j = 0$ for all $j \in \mathcal{J}(h)$ **then**
      $A_k(h) \leftarrow \varnothing$ and $\alpha_k(h) \leftarrow 0$
    else
      for all $g \in \mathcal{J}(h)$ **do**
        if $x_j = 0$ for all $j \in \mathcal{J}^g$ **then**
          $Y_k^g \leftarrow 0$
        else
          Pick $j_k^g \in \arg\min \{r_j \,:\, j \in \mathcal{J}^g, x_j > 0\}$, and set $Y_k^g \leftarrow \frac{x_{j_k^g}}{w_{j_k^g}}$
        end if
      end for
      Set $A_k(h) \leftarrow \cup_{\{g \in \mathcal{J}(h) \,:\, Y_k^g > 0\}} \bar{J}(j_k^g)$, $Y_k(h) \leftarrow \min\{Y_k^g \,:\, g \in \mathcal{J}(h), Y_k^g > 0\}$, and $\alpha_k(h) \leftarrow \frac{\tilde{v}(A_k(h)) + v_0}{\lambda_h T} Y_k(h)$
    end if
  end for
  Set $A_k \leftarrow \cup_{h \in \mathcal{H}} A_k(h)$, and $\alpha(A_k) \leftarrow \min\{\alpha_k(h) \,:\, h \in \mathcal{H}, \alpha_k(h) > 0\}$
  for all $h \in \mathcal{H}$ **do**
    if $A_k(h) \neq \varnothing$ **then**
      for all $g \in \mathcal{J}(h)$ **do**
        if $Y_k^g > 0$ **then**
          Set $x_{j_k^g} \leftarrow x_{j_k^g} - \lambda_h \alpha(A_k) T \frac{w_{j_k^g}}{\tilde{v}(A_k(h)) + v_0}$
        end if
      end for
    end if
  end for
  $k \leftarrow k + 1$
end while
Output $\boldsymbol{\alpha}$

Lemma A.11. *Given a feasible solution* $(\boldsymbol{x}, \boldsymbol{x}_0)$ *for SBLP* (2.4)*, Algorithm 1 terminates in* $O(|\mathcal{G}|n)$ *steps.*

Proof. In each iteration of the while-loop (line 2–line 28), according to the definition of Y_k^g, $Y_k(h)$, $\alpha_k(h)$, and $\alpha(A_k)$, at least one of the positive components of $\boldsymbol{x}$ is reduced to 0. Therefore, after at most n iterations of the while-loop, it holds that $\boldsymbol{x} = \mathbf{0}$ and the algorithm terminates. The for-loops in each iteration (line 3–line 16, line 18–line 26) require at most $O(|\mathcal{G}|)$ steps. (The fare classes of each itinerary g can be sorted by revenue in advance so that line 11 can be executed in constant time for each g. Also, for each $j \in \mathcal{J}$, one can compute the values of $\tilde{v}(\bar{J}(j)) = w_j + \sum_{j' \in J(j)} v_{j'}$ in advance, which can be done inductively from the highest fare class for each itinerary to the lowest fare class for the itinerary, in a total of $O(n)$ steps. Then, in line 14, $\tilde{v}(A_k(h))$ can be computed for all h in $O(|\mathcal{G}|)$ steps as follows: $\tilde{v}(A_k(h)) = \sum_{\{g \in \mathcal{J}(h)\,:\, Y_k^g > 0\}} \tilde{v}(\bar{J}(j_k^g))$.) So Algorithm 1 terminates in $O(|\mathcal{G}|n)$ steps. $\square$

Lemma A.12. *The output of Algorithm 1 satisfies the following properties:*

(1) For each iteration k*, the assortment* A_k *defined in line 17 is nested by revenue.*

(2) For each iteration k*, each market* h*, and each itinerary* $g \in \mathcal{J}(h)$*, the amount* $\lambda_h \alpha(A_k) T \frac{w_{j_k^g}}{\tilde{v}(A_k(h)) + v_0}$ *subtracted from* $x_{j_k^g}$ *is equal to the expected sales quantity of product* j_k^g *while assortment* A_k *is offered for* $\alpha(A_k)T$ *units of time.*

(3) The CDLP solution $\boldsymbol{\alpha}$ *produced by Algorithm 1 satisfies*

$$x_j \;=\; \lambda_h T \sum_{\{k\,:\, I(j, A_k) = 1\}} \alpha(A_k) P^h_{j : A_k \cap \mathcal{J}(h)}$$

for all $h \in \mathcal{H}$ *and all* $j \in \mathcal{J}(h)$*. That is, for each product* j*, the sales quantity* x_j *while* j *is the cheapest available fare class for its itinerary specified by the SBLP solution* $(\boldsymbol{x}, \boldsymbol{x}_0)$ *is equal to the sales quantity of* j *while* j *is the cheapest available fare class for its itinerary resulting from CDLP solution* $\boldsymbol{\alpha}$*.*

Proof. (1) For each k and each h, $A_k(h)$ is either $\varnothing$ or a union over itineraries g of nested by revenue assortments $\bar{J}(j_k^g)$, so each $A_k(h)$ is a nested by revenue assortment. Each A_k is a union over markets h of nested by revenue assortments $A_k(h)$, so each A_k is a nested by revenue assortment.

(2) The expected sales quantity of j_k^g while assortment A_k is offered for $\alpha(A_k)T$ units of time is equal to $\lambda_h \alpha(A_k) T P^h_{j_k^g : A_k \cap \mathcal{J}(h)} = \lambda_h \alpha(A_k) T \frac{w_{j_k^g}}{\tilde{v}(A_k \cap \mathcal{J}(h)) + v_0} = \lambda_h \alpha(A_k) T \frac{w_{j_k^g}}{\tilde{v}(A_k(h)) + v_0}$.

(3) It follows from above that

$$x_j \;=\; \sum_{\{k\,:\,I(j,A_k)=1\}} \lambda_h \alpha(A_k) T \frac{w_j}{\tilde{v}(A_k(h)) + v_0} \;=\; \lambda_h T \sum_{\{k\,:\,I(j,A_k)=1\}} \alpha(A_k) P^h_{j:A_k \cap \mathcal{J}(h)}.$$

□

Proposition A.13. *Given a feasible solution* $(\boldsymbol{x}, \boldsymbol{x}_0)$ *of SBLP* (2.4)*, Algorithm 1 computes a feasible solution* $\boldsymbol{\alpha}$ *of CDLP* (2.3)*, such that the sales quantity of each product is the same in both solutions, and the two solutions have the same objective value.*

Proof. First we show that $(\boldsymbol{x}, \boldsymbol{x}_0)$ satisfies SBLP constraint (2.4c) if and only if $\boldsymbol{\alpha}$ produced by Algorithm 1 satisfies CDLP constraint (2.3c). The left side of SBLP constraint (2.4c) is

$$\begin{aligned}
&\sum_{h \in \mathcal{H}} \sum_{j \in \mathcal{J}(h)} \left(1 + \sum_{j' \in J(j)} \frac{v_{j'}}{w_j}\right) a_f^j x_j \\
&= \sum_{h \in \mathcal{H}} \sum_{j \in \mathcal{J}(h)} \left(1 + \sum_{j' \in J(j)} \frac{v_{j'}}{w_j}\right) a_f^j \sum_{\{k\,:\,I(j,A_k)=1\}} \lambda_h \alpha(A_k) T \frac{w_j}{\tilde{v}(A_k(h)) + v_0} \\
&= \sum_k \alpha(A_k) T \sum_{h \in \mathcal{H}} \lambda_h \sum_{\{j \in \mathcal{J}(h)\,:\,I(j,A_k)=1\}} a_f^j \frac{w_j + \sum_{j' \in J(j)} v_{j'}}{\tilde{v}(A_k(h)) + v_0} \\
&= \sum_k \alpha(A_k) T \sum_{h \in \mathcal{H}} \lambda_h \sum_{\{j \in \mathcal{J}(h)\,:\,I(j,A_k)=1\}} \left(a_f^j \frac{w_j}{\tilde{v}(A_k(h)) + v_0} + \sum_{j' \in J(j)} a_f^{j'} \frac{v_{j'}}{\tilde{v}(A_k(h)) + v_0}\right) \\
&= \sum_k \alpha(A_k) T \sum_{h \in \mathcal{H}} \lambda_h \sum_{\{j \in \mathcal{J}(h)\,:\,I(j,A_k)=1\}} \left(a_f^j P^h_{j:A_k \cap \mathcal{J}(h)} + \sum_{j' \in J(j)} a_f^{j'} P^h_{j':A_k \cap \mathcal{J}(h)}\right)
\end{aligned}$$

$$= \sum_{A \subset \mathcal{J}} \alpha(A) T \sum_{h \in \mathcal{H}} \lambda_h \sum_{j \in \mathcal{J}(h)} a_f^j P^h_{j:A \cap \mathcal{J}(h)}$$
$$= \sum_{A \subset \mathcal{J}} \alpha(A) T \sum_{h \in \mathcal{H}} \lambda_h \sum_{j \in A} a_f^j P^h_{j:A \cap \mathcal{J}(h)}.$$

which is the left side of CDLP constraint (2.3c). The third equality holds because for any $j \in \mathcal{J}$ and any $j' \in J(j)$ it holds that $a_f^j = a_f^{j'}$ for all $f \in \mathcal{F}$.

Next we show that if $(\boldsymbol{x}, \boldsymbol{x}_0)$ satisfies SBLP constraints (2.4b) and (2.4d), then $\boldsymbol{\alpha}$ satisfies CDLP constraint (2.3b). Consider any h and any $g \in \mathcal{J}(h)$. In each iteration k, if there is a product $j \in \mathcal{J}^g$ such that $x_j > 0$, then one such product j_k^g is chosen. Then the quantity $x_{j_k^g}$ is reduced by $\lambda_h \alpha(A_k) T \frac{w_{j_k^g}}{\tilde{v}(A_k(h)) + v_0}$, and for all the other products $j' \in \mathcal{J}^g \setminus \{j_k^g\}$, $x_{j'}$ remains unchanged. Otherwise, if there is no product $j \in \mathcal{J}^g$ such that $x_j > 0$, then x_j remains 0 for all $j \in \mathcal{J}^g$. Thus, for any h, any $g \in \mathcal{J}(h)$, and any $j \in \mathcal{J}^g$, it holds that

$$\sum_{\{k \,:\, j_k^g = j\}} \lambda_h \alpha(A_k) T \frac{1}{\tilde{v}(A_k(h)) + v_0} \;=\; \frac{x_j}{w_j},$$

and hence

$$\sum_{\{k \,:\, Y_k^g > 0\}} \lambda_h \alpha(A_k) T \frac{1}{\tilde{v}(A_k(h)) + v_0} \;=\; \sum_{j \in \mathcal{J}^g} \sum_{\{k \,:\, j_k^g = j\}} \lambda_h \alpha(A_k) T \frac{1}{\tilde{v}(A_k(h)) + v_0}$$
$$= \sum_{j \in \mathcal{J}^g} \frac{x_j}{w_j} \;\leq\; \frac{x_0^h}{v_0},$$

where the inequality follows from SBLP constraint (2.4d), and

$$\sum_{\{k \,:\, Y_k^g > 0\}} \tilde{v}(\bar{J}(j_k^g)) \lambda_h \alpha(A_k) T \frac{1}{\tilde{v}(A_k(h)) + v_0}$$
$$= \sum_{j \in \mathcal{J}^g} \sum_{\{k \,:\, j_k^g = j\}} \tilde{v}(\bar{J}(j)) \lambda_h \alpha(A_k) T \frac{1}{\tilde{v}(A_k(h)) + v_0} \;=\; \sum_{j \in \mathcal{J}^g} \tilde{v}(\bar{J}(j)) \frac{x_j}{w_j}.$$

Note that there is at least one $g \in \mathcal{G}$ such that $Y_k^g > 0$ for all k. Let $g^* \in \mathcal{G}$ be such that

$Y_k^{g^*} > 0$ for all k, and let $h^* \in \mathcal{H}$ be such that $g^* \in \mathcal{J}(h^*)$. It follows from line 14 and line 17 of Algorithm 1 that

$$
\begin{aligned}
\sum_{A \subset \mathcal{J}} \alpha(A) \;&=\; \sum_k \alpha(A_k) \\
&= \sum_k \alpha(A_k) \frac{\lambda_{h^*} T}{\lambda_{h^*} T} \frac{\tilde{v}(A_k(h^*)) + v_0}{\tilde{v}(A_k(h^*)) + v_0} \\
&= \frac{1}{\lambda_{h^*} T} \sum_k \lambda_{h^*} \alpha(A_k) T \frac{\sum_{\{g \in \mathcal{J}(h^*)\,:\, Y_k^g > 0\}} \tilde{v}(\bar{J}(j_k^g)) + v_0}{\tilde{v}(A_k(h^*)) + v_0} \\
&= \frac{1}{\lambda_{h^*} T} \left[\sum_{g \in \mathcal{J}(h^*)} \sum_{\{k\,:\, Y_k^g > 0\}} \tilde{v}(\bar{J}(j_k^g)) \lambda_{h^*} \alpha(A_k) T \frac{1}{\tilde{v}(A_k(h^*)) + v_0} \right. \\
&\qquad \left. + \sum_k \lambda_{h^*} \alpha(A_k) T \frac{v_0}{\tilde{v}(A_k(h^*)) + v_0} \right] \\
&= \frac{1}{\lambda_{h^*} T} \left[\sum_{g \in \mathcal{J}(h^*)} \sum_{j \in \mathcal{J}^g} \tilde{v}(\bar{J}(j)) \frac{x_j}{w_j} + \sum_{\{k\,:\, Y_k^{g^*} > 0\}} \lambda_{h^*} \alpha(A_k) T \frac{v_0}{\tilde{v}(A_k(h^*)) + v_0} \right] \\
&\le \frac{1}{\lambda_{h^*} T} \left[\sum_{g \in \mathcal{J}(h^*)} \sum_{j \in \mathcal{J}^g} \left(w_j + \sum_{j' \in J(j)} v_{j'} \right) \frac{x_j}{w_j} + v_0 \frac{x_0^{h^*}}{v_0} \right] \\
&= \frac{1}{\lambda_{h^*} T} \left[\sum_{j \in \mathcal{J}(h^*)} \left(1 + \sum_{j' \in J(j)} \frac{v_{j'}}{w_j} \right) x_j + x_0^{h^*} \right] \;=\; 1,
\end{aligned}
$$

where the last equality follows from SBLP constraint (2.4b).

Next we show that the objective values of the SBLP solution $(\boldsymbol{x}, \boldsymbol{x}_0)$ and the CDLP solution $\boldsymbol{\alpha}$ are the same.

$$
\begin{aligned}
&\sum_{h \in \mathcal{H}} \sum_{j \in \mathcal{J}(h)} \left(r_j + \sum_{j' \in J(j)} r_{j'} \frac{v_{j'}}{w_j} \right) x_j \\
&= \sum_{h \in \mathcal{H}} \sum_{g \in \mathcal{J}(h)} \sum_{j \in \mathcal{J}^g} \left(r_j + \sum_{j' \in J(j)} r_{j'} \frac{v_{j'}}{w_j} \right) \sum_{\{k\,:\, j_k^g = j\}} \lambda_h \alpha(A_k) T \frac{w_j}{\tilde{v}(A_k(h)) + v_0} \\
&= \sum_k \alpha(A_k) T \sum_{h \in \mathcal{H}} \lambda_h \sum_{g \in \mathcal{J}(h)} \sum_{\{j \in \mathcal{J}^g\,:\, j_k^g = j\}} \left(r_j \frac{w_j}{\tilde{v}(A_k(h)) + v_0} + \sum_{j' \in J(j)} r_{j'} \frac{v_{j'}}{\tilde{v}(A_k(h)) + v_0} \right)
\end{aligned}
$$

$$= \sum_k \alpha(A_k) T \sum_{h \in \mathcal{H}} \lambda_h \sum_{j \in A_k \cap \mathcal{J}(h)} r_j P^h_{j:A_k \cap \mathcal{J}(h)}$$
$$= \sum_{A \subset \mathcal{J}} \alpha(A) T \sum_{h \in \mathcal{H}} \lambda_h \sum_{j \in A \cap \mathcal{J}(h)} r_j P^h_{j:A \cap \mathcal{J}(h)}.$$

Thus we have established that Algorithm 1 converts any feasible solution of SBLP (2.4) into a feasible solution of CDLP (2.3), such that the sales quantity of each product is the same in both solutions, and the two solutions have the same objective value. □

A.4.2 From CDLP to SBLP.

In this section we address the other direction: converting a CDLP solution into a SBLP solution.

Proposition A.14. *Consider any feasible solution $\boldsymbol{\alpha}$ of CDLP* (2.3) *with support on assortments that are nested by revenue. Then there is a feasible solution $(\boldsymbol{x}, \boldsymbol{x}_0)$ of SBLP* (2.4) *such that the sales quantity of each product is the same in both solutions, and the two solutions have the same objective value.*

Proof. Since $\boldsymbol{\alpha}$ has support on assortments that are nested by revenue, it follows that for any $A \subset \mathcal{J}$ such that $\alpha(A) > 0$ and for any $j \in A \cap \mathcal{J}(h)$, it holds that $J(j) \subset A \cap \mathcal{J}(h)$. For every $h \in \mathcal{H}$ and $j \in \mathcal{J}(h)$, let

$$\begin{aligned} x_j &:= \lambda_h T \sum_{\{A \subset \mathcal{J} \,:\, \mathbf{I}(j,A)=1\}} P^h_{j:A \cap \mathcal{J}(h)} \alpha(A) \\ \text{and} \quad x_0^h &:= \lambda_h T \left\{ \sum_{A \subset \mathcal{J}} P^h_{0:A \cap \mathcal{J}(h)} \alpha(A) + \left[1 - \sum_{A \subset \mathcal{J}} \alpha(A) \right] \right\}. \end{aligned}$$

Next we show that $(\boldsymbol{x}, \boldsymbol{x}_0)$ is feasible for SBLP (2.4). The balance constraint (2.4b) in the SBLP holds, since

$$x_0^h + \sum_{j \in \mathcal{J}(h)} \left(1 + \sum_{j' \in J(j)} \frac{v_{j'}}{w_j} \right) x_j$$

$$
\begin{aligned}
&= \lambda_h T \left\{ \sum_{A \subset \mathcal{J}} P^h_{0:A\cap\mathcal{J}(h)} \alpha(A) + \left[1 - \sum_{A \subset \mathcal{J}} \alpha(A) \right] \right\} \\
&\qquad + \sum_{j \in \mathcal{J}(h)} \left(1 + \sum_{j' \in J(j)} \frac{v_{j'}}{w_j} \right) \lambda_h T \sum_{\{A \subset \mathcal{J} \,:\, \mathbf{I}(j,A)=1\}} P^h_{j:A\cap\mathcal{J}(h)} \alpha(A) \\
&= \lambda_h T \Bigg\{ \sum_{A \subset \mathcal{J}} \left[P^h_{0:A\cap\mathcal{J}(h)} + \sum_{\substack{\{j \in \mathcal{J}(h):\\ \mathbf{I}(j,A)=1\}}} \left(1 + \sum_{j' \in J(j)} \frac{v_{j'}}{w_j} \right) P^h_{j:A\cap\mathcal{J}(h)} \right] \alpha(A) \\
&\qquad + \left[1 - \sum_{A \subset \mathcal{J}} \alpha(A) \right] \Bigg\} \\
&= \lambda_h T \Bigg\{ \sum_{A \subset \mathcal{J}} \left[P^h_{0:A\cap\mathcal{J}(h)} + \sum_{\substack{\{j \in \mathcal{J}(h):\\ \mathbf{I}(j,A)=1\}}} \left(P^h_{j:A\cap\mathcal{J}(h)} + \sum_{j' \in J(j)} \frac{v_{j'}}{w_j} \frac{w_j}{\tilde{v}(A \cap \mathcal{J}(h)) + v_0} \right) \right] \alpha(A) \\
&\qquad + \left[1 - \sum_{A \subset \mathcal{J}} \alpha(A) \right] \Bigg\} \\
&= \lambda_h T \Bigg\{ \sum_{A \subset \mathcal{J}} \left[P^h_{0:A\cap\mathcal{J}(h)} + \sum_{\substack{\{j \in \mathcal{J}(h):\\ \mathbf{I}(j,A)=1\}}} \left(P^h_{j:A\cap\mathcal{J}(h)} + \sum_{j' \in J(j)} P^h_{j':A\cap\mathcal{J}(h)} \right) \right] \alpha(A) \\
&\qquad + \left[1 - \sum_{A \subset \mathcal{J}} \alpha(A) \right] \Bigg\} \\
&= \lambda_h T \left\{ \sum_{A \subset \mathcal{J}} \alpha(A) + \left[1 - \sum_{A \subset \mathcal{J}} \alpha(A) \right] \right\} \; = \; \lambda T.
\end{aligned}
$$

For each $f \in \mathcal{F}$, the SBLP capacity constraint (2.4c) also holds, since

$$
\begin{aligned}
&\sum_{h \in \mathcal{H}} \sum_{j \in \mathcal{J}(h)} \left(1 + \sum_{j' \in J(j)} \frac{v_{j'}}{w_j} \right) a^j_f x_j \\
&= \sum_{h \in \mathcal{H}} \sum_{j \in \mathcal{J}(h)} \left(1 + \sum_{j' \in J(j)} \frac{v_{j'}}{w_j} \right) a^j_f \lambda_h T \sum_{\{A \subset \mathcal{J} \,:\, \mathbf{I}(j,A)=1\}} P^h_{j:A\cap\mathcal{J}(h)} \alpha(A) \\
&= \sum_{A \subset \mathcal{J}} \alpha(A) T \sum_{h \in \mathcal{H}} \lambda_h \sum_{\{j \in \mathcal{J}(h) \,:\, \mathbf{I}(j,A)=1\}} \left(1 + \sum_{j' \in J(j)} \frac{v_{j'}}{w_j} \right) a^j_f P^h_{j:A\cap\mathcal{J}(h)}
\end{aligned}
$$

$$= \sum_{A\subset\mathcal{J}} \alpha(A) T \sum_{h\in\mathcal{H}} \lambda_h \sum_{\{j\in\mathcal{J}(h)\,:\,\mathbf{I}(j,A)=1\}} a_f^j \left(P^h_{j:A\cap\mathcal{J}(h)} + \sum_{j'\in J(j)} \frac{v_{j'}}{w_j} \frac{w_j}{\tilde{v}(A\cap\mathcal{J}(h)) + v_0} \right)$$
$$= \sum_{A\subset\mathcal{J}} \alpha(A) T \sum_{h\in\mathcal{H}} \lambda_h \sum_{\{j\in\mathcal{J}(h)\,:\,\mathbf{I}(j,A)=1\}} a_f^j \left(P^h_{j:A\cap\mathcal{J}(h)} + \sum_{j'\in J(j)} P^h_{j':A\cap\mathcal{J}(h)} \right)$$
$$= \sum_{A\subset\mathcal{J}} \alpha(A) T \sum_{h\in\mathcal{H}} \lambda_h \sum_{j\in\mathcal{J}(h)} a_f^j P^h_{j:A\cap\mathcal{J}(h)} \;\leq\; c_f,$$

where the last equality holds because for any $j \in \mathcal{J}$ and any $j' \in J(j)$ it holds that $a_f^j = a_f^{j'}$ for all $f \in \mathcal{F}$. Also, for any $h \in \mathcal{H}$ and $g \in \mathcal{J}(h)$,

$$\sum_{j\in\mathcal{J}^g} \frac{x_j}{w_j} \;=\; \sum_{j\in\mathcal{J}^g} \frac{1}{w_j} \lambda_h T \sum_{\{A\subset\mathcal{J}\,:\,\mathbf{I}(j,A)=1\}} P^h_{j:A\cap\mathcal{J}(h)} \alpha(A)$$
$$= \lambda_h T \sum_{A\subset\mathcal{J}} \sum_{\{j\in\mathcal{J}^g\,:\,\mathbf{I}(j,A)=1\}} \frac{1}{w_j} \frac{w_j}{\tilde{v}(A\cap\mathcal{J}(h)) + v_0} \alpha(A)$$
$$= \lambda_h T \sum_{A\subset\mathcal{J}} \sum_{\{j\in\mathcal{J}^g\,:\,\mathbf{I}(j,A)=1\}} \frac{1}{v_0} \frac{v_0}{\tilde{v}(A\cap\mathcal{J}(h)) + v_0} \alpha(A)$$
$$= \lambda_h T \sum_{\{A\subset\mathcal{J}\,:\,A\cap\mathcal{J}^g\neq\varnothing\}} P^h_{0:A\cap\mathcal{J}(h)} \alpha(A) \frac{1}{v_0} \;\leq\; \frac{x_0^h}{v_0},$$

which is SBLP constraint (2.4d). Finally, the objective value of SBLP is

$$\sum_{h\in\mathcal{H}} \sum_{j\in\mathcal{J}(h)} \left(r_j + \sum_{j'\in J(j)} r_{j'} \frac{v_{j'}}{w_j} \right) x_j$$
$$= \sum_{h\in\mathcal{H}} \sum_{j\in\mathcal{J}(h)} \left(r_j + \sum_{j'\in J(j)} r_{j'} \frac{v_{j'}}{w_j} \right) \lambda_h T \sum_{\{A\subset\mathcal{J}\,:\,\mathbf{I}(j,A)=1\}} P^h_{j:A\cap\mathcal{J}(h)} \alpha(A)$$
$$= \sum_{A\subset\mathcal{J}} \alpha(A) T \sum_{h\in\mathcal{H}} \lambda_h \sum_{\{j\in\mathcal{J}(h)\,:\,\mathbf{I}(j,A)=1\}} \left(r_j + \sum_{j'\in J(j)} r_{j'} \frac{v_{j'}}{w_j} \right) P^h_{j:A\cap\mathcal{J}(h)}$$
$$= \sum_{A\subset\mathcal{J}} \alpha(A) T \sum_{h\in\mathcal{H}} \lambda_h \sum_{\{j\in\mathcal{J}(h)\,:\,\mathbf{I}(j,A)=1\}} \left(r_j P^h_{j:A\cap\mathcal{J}(h)} \right.$$
$$\left. + \sum_{j'\in J(j)} r_{j'} \frac{v_{j'}}{w_j} \frac{w_j}{\tilde{v}(A\cap\mathcal{J}(h)) + v_0} \right)$$

$$= \sum_{A \subset \mathcal{J}} \alpha(A) T \sum_{h \in \mathcal{H}} \lambda_h \sum_{j \in \mathcal{J}(h)} r_j P^h_{j:A \cap \mathcal{J}(h)}$$

$$= \sum_{A \subset \mathcal{J}} \alpha(A) T \sum_{h \in \mathcal{H}} \lambda_h \sum_{j \in A} r_j P^h_{j:A \cap \mathcal{J}(h)}$$

which is equal to the objective value of CDLP. □

Theorem 2.7 follows from Lemma A.11, Lemma A.12, Proposition A.13, and Proposition A.14.

A.5 Proof of Theorem 2.8

Proof of Theorem 2.8. First, we review some properties of the CDLP, and describe the associated partitioned booking limit policy. By Theorem 2.7, there is an optimal solution $\boldsymbol{\alpha}^*$ for the CDLP that is supported on a nested sequence of assortments $S_1 \supset S_2 \supset \cdots \supset S_k$, with $\alpha^*(S_i) > 0$ for $i = 1, \ldots, k$, and with each S_i nested by revenue. Let $t_0 := 0$ and $t_i := \sum_{i'=1}^{i} \alpha^*(S_{i'})T$. Thus, an optimal solution for the CDLP is to offer each assortment S_i during $(t_{i-1}, t_i]$. For the CDLP, the sales rate of product j during $(t_{i-1}, t_i]$ is given by $\lambda^i_j := \lambda P_{j:S_i}$, and the corresponding sales quantity is $\lambda^i_j(t_i - t_{i-1})$. (Note that $\lambda^i_j = 0$ if $j \notin S_i$.) It follows from (2.5) that the booking limit b^*_j of the partitioned booking limit policy satisfies

$$b^*_j = \sum_{i=1}^{k} \lambda^i_j (t_i - t_{i-1}),$$

and thus the optimal objective value z^{CDLP} of the CDLP satisfies

$$z^{\mathrm{CDLP}} = \sum_{j \in \mathcal{J}} r_j \sum_{i=1}^{k} \lambda^i_j (t_i - t_{i-1}) = \sum_{j \in \mathcal{J}} r_j b^*_j.$$

For each $i = 1, \ldots, k$ and each $j \in \mathcal{J}$, let

$$\underline{\lambda}^i_j := \min\{\lambda P_{j:S} \ : \ S_{i+1} \subseteq S \subseteq S_i, j \in S\}$$

$$\text{and} \qquad \overline{\lambda}_j^i \;:=\; \max\{\lambda P_{j:S} \;:\; S_{i+1} \subseteq S \subseteq S_i, j \in S\},$$

where $S_{k+1} := \varnothing$. Note that under the spiked-MNL model, $\underline{\lambda}_j^i > 0$ if $j \in S_i$. Also note that $\overline{\lambda}_j^i \geq \lambda P_{j:S_{i+1}} = \lambda_j^{i+1}$ for all i and all $j \in S_i$.

Given the times $0 = t_0 < t_1 < \cdots < t_k = T$ resulting from the CDLP solution, and $\varepsilon > 0$, consider the times $t_i^- := t_i - \nu_i^- \varepsilon$ and $t_i^+ := t_i + \nu_i^+ \varepsilon$, where $\nu_i^-, \nu_i^+ > 0$ are specified inductively as follows: Let $\nu_0^- = \nu_0^+ = 0$, and let $\overline{\lambda}_j^0 = 0$ for all $j \in \mathcal{J}$. Then, for each $i = 1, \ldots, k$, let

$$\nu_i^- \;:=\; \max_{j \in S_i} \left\{ \frac{\overline{\lambda}_j^{i-1} \nu_{i-1}^- + \left(\overline{\lambda}_j^{i-1} - \lambda_j^i\right) \nu_{i-1}^+ + \overline{\lambda}_j^{i-1} + \lambda_j^i}{\lambda_j^i} \right\}, \tag{A.4a}$$

$$\nu_i^+ \;:=\; \max_{j \in S_i} \left\{ \frac{\lambda_j^i \nu_{i-1}^+ + \left(\lambda_j^i - \underline{\lambda}_j^i\right) \nu_i^- + \lambda_j^i + \underline{\lambda}_j^i}{\underline{\lambda}_j^i} \right\}. \tag{A.4b}$$

Also, let $\zeta_j^0 = 0$ for all j, and for each $i = 1, \ldots, k$, and $j \in S_i$, let

$$\eta_j^i \;:=\; \lambda_j^i \left(\nu_{i-1}^+ + \nu_i^- + 1\right), \qquad \zeta_j^i \;:=\; \overline{\lambda}_j^i \left(\nu_i^- + \nu_i^+ + 1\right). \tag{A.5}$$

Note that (A.4) and (A.5) imply that

$$\lambda_j^i \left(\nu_{i-1}^+ + \nu_i^- + 1\right) \;=\; \eta_j^i \;\leq\; \underline{\lambda}_j^i \left(\nu_i^- + \nu_i^+ - 1\right), \tag{A.6a}$$

$$\overline{\lambda}_j^i \left(\nu_i^- + \nu_i^+ + 1\right) \;=\; \zeta_j^i \;\leq\; \lambda_j^{i+1} \left(\nu_i^+ + \nu_{i+1}^- - 1\right). \tag{A.6b}$$

We consider $\varepsilon > 0$ sufficiently small such that $t_{i-1}^+ < t_i^-$ for all $i = 1, \ldots, k$, and $\zeta_j^i \varepsilon < \lambda_j^{i+1}(t_{i+1} - t_i)$ for all $i = 1, \ldots, k-1$ and all $j \in S_{i+1}$. Since $t_k^+ > t_k = T$, for convenience, the analysis below considers a continuation of the booking process after time T, but we will only count the total bookings up to time T towards the total revenue. Figure 2.4 illustrates the quantities defined above.

Next we define the stochastic sales process for Poisson demand. For each $i \in \{0, \ldots, k\}$

and $j \in \mathcal{J}$, let $\hat{N}_j^{i-} := \{\hat{N}_j^{i-}(t) : t \geq 0\}$ and $\hat{N}_j^{i+} := \{\hat{N}_j^{i+}(t) : t \geq 0\}$ denote Poisson processes with rate 1, with all the Poisson processes $\{\hat{N}_j^{i\pm} : i \in \{0, \ldots, k\}, j \in \mathcal{J}\}$ independent. For any scaling factor θ, let $S^\theta(t)$ denote the assortment offered at time t under the considered policy. Then, for each product $j \in \mathcal{J}$, each $i \in \{0, \ldots, k-1\}$, and each time $t \in (t_i^+, t_{i+1}^-]$, the total sales of product j over $(0, t]$ is

$$\sum_{i'=0}^{i-1} \hat{N}_j^{i'+}\left(\theta\lambda \int_{\tau \in (t_{i'}^+, t_{i'+1}^-]} P_{j:S^\theta(\tau)} d\tau\right) + \sum_{i'=1}^{i} \hat{N}_j^{i'-}\left(\theta\lambda \int_{\tau \in (t_{i'}^-, t_{i'}^+]} P_{j:S^\theta(\tau)} d\tau\right) + \hat{N}_j^{i+}\left(\theta\lambda \int_{\tau \in (t_i^+, t]} P_{j:S^\theta(\tau)} d\tau\right),$$

and similarly, for each $i \in \{1, \ldots, k\}$, and each time $t \in (t_i^-, t_i^+]$, the total sales of product j over $(0, t]$ is

$$\sum_{i'=0}^{i-1} \hat{N}_j^{i'+}\left(\theta\lambda \int_{\tau \in (t_{i'}^+, t_{i'+1}^-]} P_{j:S^\theta(\tau)} d\tau\right) + \sum_{i'=1}^{i-1} \hat{N}_j^{i'-}\left(\theta\lambda \int_{\tau \in (t_{i'}^-, t_{i'}^+]} P_{j:S^\theta(\tau)} d\tau\right) + \hat{N}_j^{i-}\left(\theta\lambda \int_{\tau \in (t_i^-, t]} P_{j:S^\theta(\tau)} d\tau\right).$$

Note that while the assortment is S, product j is sold according to a Poisson process with rate $\theta\lambda P_{j:S}$. Let π denote a policy that, for each scaling factor θ and at each time t, prescribes the assortment $S^\theta(t)$ to be offered at time t. Then the objective value under policy π for the θ-scaled problem is given by

$$\mathsf{E}^\pi\left[\sum_{j \in \mathcal{J}} r_j \Big\{ \sum_{i=0}^{k-1} \hat{N}_j^{i+}\left(\theta\lambda \int_{\tau \in (t_i^+, t_{i+1}^-]} P_{j:S^\theta(\tau)} d\tau\right) + \sum_{i=1}^{k-1} \hat{N}_j^{i-}\left(\theta\lambda \int_{\tau \in (t_i^-, t_i^+]} P_{j:S^\theta(\tau)} d\tau\right) + \hat{N}_j^{k-}\left(\theta\lambda \int_{\tau \in (t_k^-, T]} P_{j:S^\theta(\tau)} d\tau\right) \Big\}\right].$$

Now we describe the stochastic sales process and the offered assortments under the partitioned booking limit policy. We ignore the probability 0 event that more than 1 arrival of the Poisson processes $\{\hat{N}_j^{i\pm} : i \in \{0,\dots,k\}, j \in \mathcal{J}\}$ take place at the same time. Let S_ℓ^θ denote the ℓth assortment offered under the partitioned booking limit policy (note that, except for $\ell = 1$, it does not hold in general that $S_\ell^\theta = S_\ell$), and let assortment S_ℓ^θ be offered over time period $(\tau_{\ell-1}^\theta, \tau_\ell^\theta]$. That is, $S^\theta(t) = S_\ell^\theta$ for $t \in (\tau_{\ell-1}^\theta, \tau_\ell^\theta]$. For any $i = 0,\dots,k-1$ and $\tau \in (\tau_{\ell-1}^\theta, T]$, let

$$T_{i,\ell}^{\theta+} := \begin{cases} \min\{t_{i+1}^-, \tau_\ell^\theta\} - \max\{t_i^+, \tau_{\ell-1}^\theta\} & \text{if } \tau_\ell^\theta > t_i^+ \text{ and } \tau_{\ell-1}^\theta < t_{i+1}^- \\ 0 & \text{otherwise} \end{cases}$$

$$T_{i,\ell}^{\theta+}(\tau) := \begin{cases} \min\{t_{i+1}^-, \tau\} - \max\{t_i^+, \tau_{\ell-1}^\theta\} & \text{if } \tau > t_i^+ \text{ and } \tau_{\ell-1}^\theta < t_{i+1}^- \\ 0 & \text{otherwise} \end{cases}$$

denote the duration of overlap between $(t_i^+, t_{i+1}^-]$ and $(\tau_{\ell-1}^\theta, \tau_\ell^\theta]$, and between $(t_i^+, t_{i+1}^-]$ and $(\tau_{\ell-1}^\theta, \tau]$. Similarly, for any $i = 1,\dots,k-1$ and $\tau \in (\tau_{\ell-1}^\theta, T]$, let

$$T_{i,\ell}^{\theta-} := \begin{cases} \min\{t_i^+, \tau_\ell^\theta\} - \max\{t_i^-, \tau_{\ell-1}^\theta\} & \text{if } \tau_\ell^\theta > t_i^- \text{ and } \tau_{\ell-1}^\theta < t_i^+ \\ 0 & \text{otherwise} \end{cases}$$

$$T_{i,\ell}^{\theta-}(\tau) := \begin{cases} \min\{t_i^+, \tau\} - \max\{t_i^-, \tau_{\ell-1}^\theta\} & \text{if } \tau > t_i^- \text{ and } \tau_{\ell-1}^\theta < t_i^+ \\ 0 & \text{otherwise} \end{cases}$$

denote the duration of overlap between $(t_i^-, t_i^+]$ and $(\tau_{\ell-1}^\theta, \tau_\ell^\theta]$, and between $(t_i^-, t_i^+]$ and $(\tau_{\ell-1}^\theta, \tau]$, and let

$$T_{k,\ell}^{\theta-} := \begin{cases} \tau_\ell^\theta - \max\{t_k^-, \tau_{\ell-1}^\theta\} & \text{if } \tau_\ell^\theta > t_k^- \\ 0 & \text{otherwise} \end{cases}$$

$$T_{k,\ell}^{\theta-}(\tau) := \begin{cases} \tau - \max\{t_k^-, \tau_{\ell-1}^\theta\} & \text{if } \tau > t_k^- \\ 0 & \text{otherwise} \end{cases}$$

denote the duration of overlap between $(t_k^-, T]$ and $(\tau_{\ell-1}^\theta, \tau_\ell^\theta]$, and between $(t_k^-, T]$ and $(\tau_{\ell-1}^\theta, \tau]$. Specifically, let $\tau_0^\theta := 0$, and let $S_1^\theta := S_1$ denote the first assortment under the partitioned booking limit policy. For each $\ell \in \{1, 2, \ldots\}$ such that $\tau_{\ell-1}^\theta < T$ and $S_\ell^\theta \neq \varnothing$, let

$$\begin{aligned} \tau_\ell^\theta \;:=\; & \min\Big\{T,\ \min\Big\{ \inf\Big\{\tau \in (\tau_{\ell-1}^\theta, T] \;: \\ & \sum_{\ell'=1}^{\ell-1}\left[\sum_{i=0}^{k-1}\hat{N}_j^{i+}\left(\theta\lambda P_{j:S_{\ell'}^\theta}T_{i,\ell'}^{\theta+}\right) + \sum_{i=1}^{k}\hat{N}_j^{i-}\left(\theta\lambda P_{j:S_{\ell'}^\theta}T_{i,\ell'}^{\theta-}\right)\right] \\ & + \left[\sum_{i=0}^{k-1}\hat{N}_j^{i+}\left(\theta\lambda P_{j:S_\ell^\theta}T_{i,\ell}^{\theta+}(\tau)\right) + \sum_{i=1}^{k}\hat{N}_j^{i-}\left(\theta\lambda P_{j:S_\ell^\theta}T_{i,\ell}^{\theta-}(\tau)\right)\right] \;=\; \theta b_j^*\Big\} \;:\; j \in S_\ell^\theta\Big\}\Big\} \end{aligned}$$

denote the last time that assortment S_ℓ^θ is offered (with the convention that $\inf \varnothing = \infty$). If $\tau_\ell^\theta < T$, then let j_ℓ^θ be the (unique) $j \in S_\ell^\theta$ such that

$$\begin{aligned} & \sum_{\ell'=1}^{\ell-1}\left[\sum_{i=0}^{k-1}\hat{N}_j^{i+}\left(\theta\lambda P_{j:S_{\ell'}^\theta}T_{i,\ell'}^{\theta+}\right) \;+\; \sum_{i=1}^{k}\hat{N}_j^{i-}\left(\theta\lambda P_{j:S_{\ell'}^\theta}T_{i,\ell'}^{\theta-}\right)\right] \\ & \qquad + \;\left[\sum_{i=0}^{k-1}\hat{N}_j^{i+}\left(\theta\lambda P_{j:S_\ell^\theta}T_{i,\ell}^{\theta+}(\tau_\ell^\theta)\right) \;+\; \sum_{i=1}^{k}\hat{N}_j^{i-}\left(\theta\lambda P_{j:S_\ell^\theta}T_{i,\ell}^{\theta-}(\tau_\ell^\theta)\right)\right] \;=\; \theta b_j^*, \end{aligned}$$

that is, j_ℓ^θ denotes the first product in S_ℓ^θ that sells out. Then $S_{\ell+1}^\theta := S_\ell^\theta \setminus \{j_\ell^\theta\}$.

Let $N_j^\theta(t)$ denote the quantity of product j sold up to time t under the partitioned booking limit policy, that is, for $t \in (\tau_{\ell-1}^\theta, \tau_\ell^\theta]$,

$$\begin{aligned} N_j^\theta(t) \;:=\; & \sum_{\ell'=1}^{\ell-1}\left[\sum_{i=0}^{k-1}\hat{N}_j^{i+}\left(\theta\lambda P_{j:S_{\ell'}^\theta}T_{i,\ell'}^{\theta+}\right) \;+\; \sum_{i=1}^{k}\hat{N}_j^{i-}\left(\theta\lambda P_{j:S_{\ell'}^\theta}T_{i,\ell'}^{\theta-}\right)\right] \\ & \qquad + \;\left[\sum_{i=0}^{k-1}\hat{N}_j^{i+}\left(\theta\lambda P_{j:S_\ell^\theta}T_{i,\ell}^{\theta+}(t)\right) \;+\; \sum_{i=1}^{k}\hat{N}_j^{i-}\left(\theta\lambda P_{j:S_\ell^\theta}T_{i,\ell}^{\theta-}(t)\right)\right] \end{aligned}$$

Thus, if $\tau_\ell^\theta < T$, then τ_ℓ^θ satisfies

$$\begin{aligned}\tau_\ell^\theta \;&=\; \min\left\{\inf\left\{\tau > \tau_{\ell-1}^\theta \;:\; N_j^\theta(\tau) \;=\; \theta b_j^*\right\} \;:\; j \in S_\ell^\theta\right\}\\ &=\; \inf\left\{\tau > \tau_{\ell-1}^\theta \;:\; N_{j_\ell^\theta}^\theta(\tau) \;=\; \theta b_{j_\ell^\theta}^*\right\}.\end{aligned}$$

Next we define the following events for each θ and each $i = 1, \ldots, k$:

$$\begin{aligned}E_i^{\theta-} &:= \bigcap_{j\in S_i}\left\{\sum_{i'=1}^{i}\theta\lambda_j^{i'}(t_{i'}-t_{i'-1}) - \theta\eta_j^i\varepsilon \;<\; N_j^\theta(t_i^-) \;<\; \sum_{i'=1}^{i}\theta\lambda_j^{i'}(t_{i'}-t_{i'-1})\right\}\\ E_i^{\theta+} &:= \bigcap_{j\in S_i}\left\{\sum_{i'=1}^{i}\theta\lambda_j^{i'}(t_{i'}-t_{i'-1}) \;\le\; N_j^\theta(t_i^+) \;\le\; \sum_{i'=1}^{i}\theta\lambda_j^{i'}(t_{i'}-t_{i'-1}) + \theta\zeta_j^i\varepsilon\right\}\\ F_i^{\theta+} &:= \cap_{i'=1}^{i}\left(E_{i'}^{\theta-}\cap E_{i'}^{\theta+}\right)\\ F_i^{\theta-} &:= F_{i-1}^{\theta+}\cap E_i^{\theta-}\end{aligned}$$

Similarly, let $F_0^{\theta+} = E_0^{\theta+} := \cap_{j\in\mathcal{J}}\left\{0 \le N_j^\theta(t_0^+) \le \theta\zeta_j^0\varepsilon\right\}$. That is, $E_i^{\theta-}$ is the event that for all products in S_i, the booking quantity at time t_i^- is slightly below the CDLP quantity at time t_i, and $E_i^{\theta+}$ is the event that for all products in S_i, the booking quantity at time t_i^+ is equal to or slightly above the CDLP quantity at time t_i. Note that, because the booking limit of product $j \in S_i \setminus S_{i+1}$ is $\theta b_j^* = \theta\sum_{i'=1}^{i}\lambda_j^{i'}(t_{i'}-t_{i'-1})$, and the booking limit of product $j \in S_{i+1}$ is at least $\theta\sum_{i'=1}^{i+1}\lambda_j^{i'}(t_{i'}-t_{i'-1})$, the event $F_{i+1}^{\theta-} := F_i^{\theta+}\cap E_{i+1}^{\theta-}$ implies that $S^\theta(t) = S_{i+1}$ for all $t \in (t_i^+, t_{i+1}^-]$; i.e., the assortment S_{i+1} is offered during $(t_i^+, t_{i+1}^-]$.

Lemma A.15. *For all $i = 0, \ldots, k-1$, the following holds:*

$$\begin{aligned}&F_i^{\theta+}\bigcap_{j\in S_{i+1}}\left\{\theta\lambda_j^{i+1}(t_{i+1}-t_i) - \theta\eta_j^{i+1}\varepsilon < \hat{N}_j^{i+}(\theta\lambda_j^{i+1}(t_{i+1}^- - t_i^+)) < \theta\lambda_j^{i+1}(t_{i+1}-t_i) - \theta\zeta_j^i\varepsilon\right\}\\ &\quad\subset\; F_i^{\theta+}\bigcap_{j\in S_{i+1}}\left\{\sum_{i'=1}^{i+1}\theta\lambda_j^{i'}(t_{i'}-t_{i'-1}) - \theta\eta_j^{i+1}\varepsilon \;<\; N_j^\theta(t_{i+1}^-) \;<\; \sum_{i'=1}^{i+1}\theta\lambda_j^{i'}(t_{i'}-t_{i'-1})\right\}.\end{aligned}$$

Proof. Consider any sample path in the event on the left. We show that the sample path is in the event on the right. Recall that since the sample path is in $F_i^{\theta+}$, it holds that

$S^\theta(t_i^+) = S_{i+1}$. Let ℓ be such that $S^\theta(t_i^+) = S_{i+1} = S_\ell^\theta$.

We show by contradiction that, for the considered sample path, no product can reach its booking limit during $[t_i^+, t_{i+1}^-)$. Suppose that $\tau_\ell^\theta \in [t_i^+, t_{i+1}^-)$. Thus $j_\ell^\theta \in S_{i+1}$ satisfies $N_{j_\ell^\theta}^\theta(\tau_\ell^\theta) = \theta b_{j_\ell^\theta}^*$. Then

$$\begin{aligned}
N_{j_\ell^\theta}^\theta(\tau_\ell^\theta) &= N_{j_\ell^\theta}^\theta(t_i^+) + \hat{N}_{j_\ell^\theta}^{i+}(\theta\lambda_{j_\ell^\theta}^{i+1}(\tau_\ell^\theta - t_i^+)) \\
&\le \left(\sum_{i'=1}^{i} \theta\lambda_{j_\ell^\theta}^{i'}(t_{i'} - t_{i'-1}) + \theta\zeta_{j_\ell^\theta}^{i}\varepsilon\right) + \hat{N}_{j_\ell^\theta}^{i+}(\theta\lambda_{j_\ell^\theta}^{i+1}(\tau_\ell^\theta - t_i^+)) \\
&\le \left(\sum_{i'=1}^{i} \theta\lambda_{j_\ell^\theta}^{i'}(t_{i'} - t_{i'-1}) + \theta\zeta_{j_\ell^\theta}^{i}\varepsilon\right) + \hat{N}_{j_\ell^\theta}^{i+}(\theta\lambda_{j_\ell^\theta}^{i+1}(t_{i+1}^- - t_i^+)) \\
&< \left(\sum_{i'=1}^{i} \theta\lambda_{j_\ell^\theta}^{i'}(t_{i'} - t_{i'-1}) + \theta\zeta_{j_\ell^\theta}^{i}\varepsilon\right) + \left(\theta\lambda_{j_\ell^\theta}^{i+1}(t_{i+1} - t_i) - \theta\zeta_{j_\ell^\theta}^{i}\varepsilon\right) \;\le\; \theta b_{j_\ell^\theta}^*.
\end{aligned}$$

The first inequality holds because the sample path is in $F_i^{\theta+}$, the second inequality holds because $\tau_\ell^\theta < t_{i+1}^-$, the third inequality holds because the sample path is in the event on the left, and the fourth inequality holds because $j_\ell^\theta \in S_{i+1}$. This contradicts $N_{j_\ell^\theta}^\theta(\tau_\ell^\theta) = \theta b_{j_\ell^\theta}^*$. Thus $S^\theta(t) = S_{i+1}$ for all $t \in [t_i^+, t_{i+1}^-]$.

Thus, for each $j \in S_{i+1}$ it holds that

$$\begin{aligned}
N_j^\theta(t_{i+1}^-) \;&=\; N_j^\theta(t_i^+) + \hat{N}_j^{i+}(\theta\lambda_j^{i+1}(t_{i+1}^- - t_i^+)) \\
\ge\; &\sum_{i'=1}^{i} \theta\lambda_j^{i'}(t_{i'} - t_{i'-1}) + \hat{N}_j^{i+}(\theta\lambda_j^{i+1}(t_{i+1}^- - t_i^+)) \;>\; \sum_{i'=1}^{i+1} \theta\lambda_j^{i'}(t_{i'} - t_{i'-1}) - \theta\eta_j^{i+1}\varepsilon.
\end{aligned}$$

The inequalities hold because the sample path is in the event on the left. Also,

$$\begin{aligned}
N_j^\theta(t_{i+1}^-) \;&=\; N_j^\theta(t_i^+) + \hat{N}_j^{i+}(\theta\lambda_j^{i+1}(t_{i+1}^- - t_i^+)) \\
&\le\; \left(\sum_{i'=1}^{i} \theta\lambda_j^{i'}(t_{i'} - t_{i'-1}) + \theta\zeta_j^{i}\varepsilon\right) + \hat{N}_j^{i+}(\theta\lambda_j^{i+1}(t_{i+1}^- - t_i^+)) \\
&<\; \left(\sum_{i'=1}^{i} \theta\lambda_j^{i'}(t_{i'} - t_{i'-1}) + \theta\zeta_j^{i}\varepsilon\right) + \left(\theta\lambda_j^{i+1}(t_{i+1} - t_i) - \theta\zeta_j^{i}\varepsilon\right)
\end{aligned}$$

$$= \sum_{i'=1}^{i+1} \theta\lambda_j^{i'}(t_{i'} - t_{i'-1}).$$

Therefore the sample path is in the event on the right. $\square$

Lemma A.16. *For all $i = 1, \ldots, k$, the following holds:*

$$\begin{aligned}
& F_i^{\theta-} \bigcap_{j\in S_i} \left\{\theta\eta_j^i\varepsilon \;\leq\; \hat{N}_j^{i-}\left(\theta\underline{\lambda}_j^i(t_i^+ - t_i^-)\right)\right\} \bigcap_{j\in S_{i+1}} \left\{\hat{N}_j^{i-}\left(\theta\overline{\lambda}_j^i(t_i^+ - t_i^-)\right) \;\leq\; \theta\zeta_j^i\varepsilon\right\} \\
\subset\; & F_i^{\theta-} \bigcap_{j\in S_i\setminus S_{i+1}} \left\{\sum_{i'=1}^{i} \theta\lambda_j^{i'}(t_{i'} - t_{i'-1}) - N_j^\theta(t_i^-) \;=\; \hat{N}_j^{i-}\left(\theta\lambda\int_{\tau\in(t_i^-,t_i^+]} P_{j:S^\theta(\tau)}d\tau\right)\right\} \\
& \bigcap_{j\in S_{i+1}} \left\{\theta\eta_j^i\varepsilon \;\leq\; \hat{N}_j^{i-}\left(\theta\lambda\int_{\tau\in(t_i^-,t_i^+]} P_{j:S^\theta(\tau)}d\tau\right) \;\leq\; \theta\zeta_j^i\varepsilon\right\}.
\end{aligned}$$

Proof. Consider any sample path in the event on the left. We show that the sample path is in the event on the right. Recall that since the sample path is in $F_i^{\theta-}$, it holds that $S^\theta(t_i^-) = S_i$. Let ℓ be such that $S^\theta(t_i^-) = S_i = S_\ell^\theta$.

First we show by contradiction that it cannot hold that $S^\theta(t) = S_i$ for all $t \in (t_i^-, t_i^+]$, that is, at least one product $j \in S_i$ must reach its booking limit during $(t_i^-, t_i^+]$. Suppose that $S^\theta(t) = S_i$ for all $t \in (t_i^-, t_i^+]$. Consider any $j \in S_i \setminus S_{i+1}$. Then

$$\begin{aligned}
& N_j^\theta(t_i^+) \;=\; N_j^\theta(t_i^-) + \hat{N}_j^{i-}\left(\theta\lambda P_{j:S_i}(t_i^+ - t_i^-)\right) \\
\geq\; & N_j^\theta(t_i^-) + \hat{N}_j^{i-}\left(\theta\underline{\lambda}_j^i(t_i^+ - t_i^-)\right) \;>\; \left(\sum_{i'=1}^{i} \theta\lambda_j^{i'}(t_{i'} - t_{i'-1}) - \theta\eta_j^i\varepsilon\right) + \theta\eta_j^i\varepsilon \;=\; \theta b_j^*.
\end{aligned}$$

The first inequality follows because the definition of $\underline{\lambda}_j^i$ implies that $\lambda P_{j:S_i} \geq \underline{\lambda}_j^i$, and the second inequality follows from the definition of the event on the left. Under the partitioned booking limit policy it cannot hold that $N_j^\theta(t_i^+) > \theta b_j^*$, and therefore for any sample path in the event on the left it cannot hold that $S^\theta(t) = S_i$ for all $t \in (t_i^-, t_i^+]$. Thus, $\tau_\ell^\theta \in [t_i^-, t_i^+)$.

Next we show that $j_\ell^\theta \notin S_{i+1}$, that is, a product $j \in S_{i+1}$ cannot be the first to reach its booking limit during $(t_i^-, t_i^+]$. Consider any $j \in S_{i+1}$. Then

$$
\begin{aligned}
N_j^\theta(\tau_\ell^\theta) &= N_j^\theta(t_i^-) + \hat{N}_j^{i-}\left(\theta\lambda P_{j:S_i}(\tau_\ell^\theta - t_i^-)\right) \\
&\leq N_j^\theta(t_i^-) + \hat{N}_j^{i-}\left(\theta\overline{\lambda}_j^i(\tau_\ell^\theta - t_i^-)\right) \\
&\leq N_j^\theta(t_i^-) + \hat{N}_j^{i-}\left(\theta\lambda P_{j:S_i}(t_i^+ - t_i^-)\right) \\
&< \sum_{i'=1}^{i} \theta\lambda_j^{i'}(t_{i'} - t_{i'-1}) + \theta\zeta_j^i\varepsilon \\
&< \sum_{i'=1}^{i} \theta\lambda_j^{i'}(t_{i'} - t_{i'-1}) + \theta\lambda_j^{i+1}\left(t_{i+1} - t_i\right) \;\leq\; \theta b_j^*.
\end{aligned}
$$

The first inequality follows from the definition of $\overline{\lambda}_j^i$, the second inequality follows since $\tau_\ell^\theta < t_i^+$, the third inequality follows from the definition of the event on the left, the fourth inequality follows from the assumption that $\varepsilon > 0$ is sufficiently small such that $\zeta_j^i\varepsilon < \lambda_j^{i+1}\left(t_{i+1} - t_i\right)$ for all $i = 1, \ldots, k-1$ and all $j \in S_{i+1}$, and the fifth inequality follows from $j \in S_{i+1}$. Thus $j_\ell^\theta \in S_i \setminus S_{i+1}$, and hence $S_{\ell+1}^\theta = S_\ell^\theta \setminus \{j_\ell^\theta\}$ satisfies $S_{i+1} \subset S_{\ell+1}^\theta \subset S_i$.

Next, we continue by induction on ℓ. Suppose that for some $\tilde{\ell} \geq \ell$ it holds that $\tau_{\tilde{\ell}}^\theta < t_i^+$ and $S_{i+1} \subset S_{\tilde{\ell}+1}^\theta \subset S_i$. We consider two cases: either $S_{\tilde{\ell}+1}^\theta \setminus S_{i+1} \neq \varnothing$ or $S_{\tilde{\ell}+1}^\theta \setminus S_{i+1} = \varnothing$. Case $S_{\tilde{\ell}+1}^\theta \setminus S_{i+1} \neq \varnothing$: In this case we repeat the argument above. First we show by contradiction that it cannot hold that $S^\theta(t) = S_{\tilde{\ell}+1}^\theta$ for all $t \in (\tau_{\tilde{\ell}}^\theta, t_i^+]$, that is, at least one product $j \in S_{\tilde{\ell}+1}^\theta$ must reach its booking limit during $(\tau_{\tilde{\ell}}^\theta, t_i^+]$. Suppose that $S^\theta(t) = S_{\tilde{\ell}+1}^\theta$ for all $t \in (\tau_{\tilde{\ell}}^\theta, t_i^+]$. Consider any $j \in S_{\tilde{\ell}+1}^\theta \setminus S_{i+1}$. Note that $j \in S_i \setminus S_{i+1}$. Then

$$
\begin{aligned}
N_j^\theta(t_i^+) &= N_j^\theta(t_i^-) + \hat{N}_j^{i-}\left(\theta\lambda P_{j:S_i}(\tau_\ell^\theta - t_i^-)\right) \\
&\qquad + \sum_{\ell'=\ell+1}^{\tilde{\ell}} \hat{N}_j^{i-}\left(\theta\lambda P_{j:S_{\ell'}^\theta}(\tau_{\ell'}^\theta - \tau_{\ell'-1}^\theta)\right) + \hat{N}_j^{i-}\left(\theta\lambda P_{j:S_{\tilde{\ell}+1}^\theta}(t_i^+ - \tau_{\tilde{\ell}}^\theta)\right) \\
&\geq N_j^\theta(t_i^-) + \hat{N}_j^{i-}\left(\theta\underline{\lambda}_j^i(t_i^+ - t_i^-)\right) \\
&> \left(\sum_{i'=1}^{i} \theta\lambda_j^{i'}(t_{i'} - t_{i'-1}) - \theta\eta_j^i\varepsilon\right) + \theta\eta_j^i\varepsilon \;=\; \theta b_j^*.
\end{aligned}
$$

The first inequality holds because $S_{i+1} \subset S^\theta_{\ell'} \subset S_i$ and thus $\lambda P_{j:S^\theta_{\ell'}} \geq \underline{\lambda}^i_j$ for all $\ell' = \ell, \ldots, \tilde{\ell}+1$, and the second inequality follows from the definition of the event on the left. Under the partitioned booking limit policy it cannot hold that $N^\theta_j(t^+_i) > \theta b^*_j$, and therefore it cannot hold that $S^\theta(t) = S^\theta_{\tilde{\ell}+1}$ for all $t \in (\tau^\theta_{\tilde{\ell}}, t^+_i]$. Thus, $\tau^\theta_{\tilde{\ell}+1} \in (\tau^\theta_{\tilde{\ell}}, t^+_i)$.

Next we show that $j^\theta_{\tilde{\ell}+1} \notin S_{i+1}$, that is, a product $j \in S_{i+1}$ cannot be the first to reach its booking limit during $(\tau^\theta_{\tilde{\ell}}, t^+_i]$. Consider any $j \in S_{i+1}$. Then

$$\begin{aligned}
N^\theta_j(\tau^\theta_{\tilde{\ell}+1}) &= N^\theta_j(t^-_i) + \hat{N}^{i-}_j\left(\theta\lambda P_{j:S_i}(\tau^\theta_\ell - t^-_i)\right) + \sum_{\ell'=\ell+1}^{\tilde{\ell}+1} \hat{N}^{i-}_j\left(\theta\lambda P_{j:S^\theta_{\ell'}}(\tau^\theta_{\ell'} - \tau^\theta_{\ell'-1})\right) \\
&\leq N^\theta_j(t^-_i) + \hat{N}^{i-}_j\left(\theta\overline{\lambda}^i_j(\tau^\theta_{\tilde{\ell}+1} - t^-_i)\right) \\
&\leq N^\theta_j(t^-_i) + \hat{N}^{i-}_j\left(\theta\overline{\lambda}^i_j(t^+_i - t^-_i)\right) \\
&< \sum_{i'=1}^{i} \theta\lambda^{i'}_j(t_{i'} - t_{i'-1}) + \theta\zeta^i_j\varepsilon \\
&< \sum_{i'=1}^{i} \theta\lambda^{i'}_j(t_{i'} - t_{i'-1}) + \theta\lambda^{i+1}_j(t_{i+1} - t_i) \;\leq\; \theta b^*_j.
\end{aligned}$$

The first inequality holds because $S_{i+1} \subset S^\theta_{\ell'} \subset S_i$ and thus $\lambda P_{j:S^\theta_{\ell'}} \leq \overline{\lambda}^i_j$ for all $\ell' = \ell, \ldots, \tilde{\ell}+1$, the second inequality holds since $\tau^\theta_{\tilde{\ell}+1} < t^+_i$, the third inequality follows from the definition of the event on the left, the fourth inequality follows from the assumption that $\varepsilon > 0$ is sufficiently small such that $\zeta^i_j\varepsilon < \lambda^{i+1}_j(t_{i+1} - t_i)$ for all $i = 1, \ldots, k-1$ and all $j \in S_{i+1}$, and the fifth inequality follows from $j \in S_{i+1}$. Thus $j^\theta_{\tilde{\ell}+1} \in S^\theta_{\tilde{\ell}+1} \setminus S_{i+1}$, and hence $S^\theta_{\tilde{\ell}+2} = S^\theta_{\tilde{\ell}+1} \setminus \{j^\theta_{\tilde{\ell}+1}\}$ satisfies $S_{i+1} \subset S^\theta_{\tilde{\ell}+2} \subset S_i$. Hence, in the case $S^\theta_{\tilde{\ell}+1} \setminus S_{i+1} \neq \varnothing$, the induction continues.

Case $S^\theta_{\tilde{\ell}+1} \setminus S_{i+1} = \varnothing$: Then $S^\theta_{\tilde{\ell}+1} = S_{i+1}$. That is, for each $j \in S_i \setminus S_{i+1}$, it holds that $N^\theta_j(t^+_i) = N^\theta_j(\tau^\theta_{\tilde{\ell}}) = \theta b^*_j = \sum_{i'=1}^{i} \theta\lambda^{i'}_j(t_{i'} - t_{i'-1})$, and thus

$$N^\theta_j(t^-_i) + \hat{N}^{i-}_j\left(\theta\lambda\int_{\tau\in(t^-_i, t^+_i]} P_{j:S^\theta(\tau)}d\tau\right) = \sum_{i'=1}^{i} \theta\lambda^{i'}_j(t_{i'} - t_{i'-1}).$$

Next we show by contradiction that $S^\theta(t) = S_{i+1}$ for all $t \in (\tau^\theta_{\tilde{\ell}}, t_i^+]$. Suppose that $\tau^\theta_{\tilde{\ell}+1} < t_i^+$. Thus $j^\theta_{\tilde{\ell}+1} \in S_{i+1}$ satisfies $N^\theta_{j^\theta_{\tilde{\ell}+1}}(\tau^\theta_{\tilde{\ell}+1}) = \theta b^*_{j^\theta_{\tilde{\ell}+1}}$. Then

$$\begin{aligned}
N^\theta_{j^\theta_{\tilde{\ell}+1}}(\tau^\theta_{\tilde{\ell}+1}) &= N^\theta_{j^\theta_{\tilde{\ell}+1}}(t_i^-) + \hat{N}^{i-}_{j^\theta_{\tilde{\ell}+1}}\left(\theta\lambda P_{j^\theta_{\tilde{\ell}+1}:S_i}(\tau^\theta_\ell - t_i^-)\right) \\
&\quad + \sum_{\ell'=\ell+1}^{\tilde{\ell}+1} \hat{N}^{i-}_{j^\theta_{\tilde{\ell}+1}}\left(\theta\lambda P_{j^\theta_{\tilde{\ell}+1}:S^\theta_{\ell'}}(\tau^\theta_{\ell'} - \tau^\theta_{\ell'-1})\right) \\
&\le N^\theta_{j^\theta_{\tilde{\ell}+1}}(t_i^-) + \hat{N}^{i-}_{j^\theta_{\tilde{\ell}+1}}\left(\theta\overline{\lambda}^i_{j^\theta_{\tilde{\ell}+1}}(\tau^\theta_{\tilde{\ell}+1} - t_i^-)\right) \\
&\le N^\theta_{j^\theta_{\tilde{\ell}+1}}(t_i^-) + \hat{N}^{i-}_{j^\theta_{\tilde{\ell}+1}}\left(\theta\overline{\lambda}^i_{j^\theta_{\tilde{\ell}+1}}(t_i^+ - t_i^-)\right) \\
&< \sum_{i'=1}^{i} \theta\lambda^{i'}_{j^\theta_{\tilde{\ell}+1}}(t_{i'} - t_{i'-1}) + \theta\zeta^i_{j^\theta_{\tilde{\ell}+1}}\varepsilon \\
&< \sum_{i'=1}^{i} \theta\lambda^{i'}_{j^\theta_{\tilde{\ell}+1}}(t_{i'} - t_{i'-1}) + \theta\lambda^{i+1}_{j^\theta_{\tilde{\ell}+1}}(t_{i+1} - t_i) \;\le\; \theta b^*_{j^\theta_{\tilde{\ell}+1}}.
\end{aligned}$$

The first inequality holds because $S_{i+1} \subset S^\theta_{\ell'} \subset S_i$ and thus $\lambda P_{j^\theta_{\tilde{\ell}+1}:S^\theta_{\ell'}} \le \overline{\lambda}^i_{j^\theta_{\tilde{\ell}+1}}$ for all $\ell' = \ell, \ldots, \tilde{\ell}+1$, the second inequality holds since $\tau^\theta_{\tilde{\ell}+1} < t_i^+$, the third inequality follows from the definition of the event on the left, the fourth inequality follows from the assumption that $\varepsilon > 0$ is sufficiently small such that $\zeta^i_j\varepsilon < \lambda^{i+1}_j(t_{i+1} - t_i)$ for all $i = 1, \ldots, k-1$ and all $j \in S_{i+1}$, and the fifth inequality follows from $j^\theta_{\tilde{\ell}+1} \in S_{i+1}$. This contradicts $N^\theta_{j^\theta_{\tilde{\ell}+1}}(\tau^\theta_{\tilde{\ell}+1}) = \theta b^*_{j^\theta_{\tilde{\ell}+1}}$.

Thus, for each $\tau \in [t_i^-, t_i^+]$ it holds that $S_{i+1} \subset S^\theta(\tau) \subset S_i$. Hence, for each $j \in S_{i+1}$, it holds that $\hat{N}^{i-}_j\left(\theta\underline{\lambda}^i_j(t_i^+ - t_i^-)\right) \le \hat{N}^{i-}_j\left(\theta\lambda\int_{\tau\in(t_i^-,t_i^+]} P_{j:S^\theta(\tau)}d\tau\right) \le \hat{N}^{i-}_j\left(\theta\overline{\lambda}^i_j(t_i^+ - t_i^-)\right)$. Thus it follows from the event on the left that $\theta\eta^i_j\varepsilon \le \hat{N}^{i-}_j\left(\theta\lambda\int_{\tau\in(t_i^-,t_i^+]} P_{j:S^\theta(\tau)}d\tau\right) \le \theta\zeta^i_j\varepsilon$. Thereby it has been established that the sample path is in the event on the right. $\square$

We will use the following concentration inequality for Poisson process [104]:

Lemma A.17 (Poisson tail bound). *Let $\{N(t), t \ge 0\}$ be a Poisson process with unit rate.*

For any $x > 0$, $t > 0$, $\varepsilon > 0$, it holds that

$$\begin{aligned}
\mathsf{P}\left[N(xt) - xt \geq x\varepsilon\right] \; &\leq \; \exp\big(- xth(\varepsilon/t)\big), \\
\mathsf{P}\left[N(xt) - xt \leq -x\varepsilon\right] \; &\leq \; \exp\big(- xth(\varepsilon/t)\big),
\end{aligned}$$

where $h(y) := (1+y)\log(1+y) - y$. (Note that $h(y) > 0$ for all $y > 0$.)

For each $i = 1, \ldots, k$, let

$$\begin{aligned}
\delta_i^{\theta-} \; &:= \; 2\sum_{j \in S_i} \exp\left(-\theta\lambda_j^i (t_i^- - t_{i-1}^+)\, h\left(\frac{\varepsilon}{t_i^- - t_{i-1}^+}\right)\right), \\
\delta_i^{\theta+} \; &:= \; \sum_{j \in S_i} \exp\left(-\theta\underline{\lambda}_j^i (t_i^+ - t_i^-)\, h\left(\frac{\varepsilon}{t_i^+ - t_i^-}\right)\right) \\
&\qquad\qquad + \sum_{j \in S_i} \exp\left(-\theta\overline{\lambda}_j^i (t_i^+ - t_i^-)\, h\left(\frac{\varepsilon}{t_i^+ - t_i^-}\right)\right).
\end{aligned}$$

Next, we prove by induction on i that

$$\mathsf{P}\left[F_i^{\theta+}\right] \; \geq \; 1 - \sum_{i'=1}^{i} \left(\delta_{i'}^{\theta-} + \delta_{i'}^{\theta+}\right). \tag{A.7}$$

Base Case: $i = 0$. Since $t_0^+ = 0$ and $\zeta_j^0 = 0$ for all $j \in \mathcal{J}$, it holds that $0 \leq N_j^\theta(t_0^+) \leq \theta\zeta_j^0$, so $\mathsf{P}[F_0^{\theta+}] = 1$.

Induction Step: from i to $i+1$. Suppose the result holds for $i \in \{0, \ldots, k-1\}$. Then

$$\begin{aligned}
&\mathsf{P}\left[F_{i+1}^{\theta-} \mid F_i^{\theta+}\right] \; = \; \mathsf{P}\left[E_{i+1}^{\theta-} \mid F_i^{\theta+}\right] \\
&= \; \mathsf{P}\Big[\bigcap_{j \in S_{i+1}} \Big\{ \sum_{i'=1}^{i+1} \theta\lambda_j^{i'}(t_{i'} - t_{i'-1}) - \theta\eta_j^{i+1}\varepsilon \; < \; N_j^\theta(t_{i+1}^-) \; < \; \sum_{i'=1}^{i+1} \theta\lambda_j^{i'}(t_{i'} - t_{i'-1})\Big\} \mid F_i^{\theta+}\Big] \\
&\geq \; \mathsf{P}\Big[\bigcap_{j \in S_{i+1}} \Big\{\theta\lambda_j^{i+1}(t_{i+1} - t_i) - \theta\eta_j^{i+1}\varepsilon \; < \; \hat{N}_j^{i+}(\theta\lambda_j^{i+1}(t_{i+1}^- - t_i^+)) \; < \; \theta\lambda_j^{i+1}(t_{i+1} - t_i) - \theta\zeta_j^i\varepsilon\Big\}\Big] \\
&= \; 1 - \mathsf{P}\Big[\bigcup_{j \in S_{i+1}} \Big(\Big\{\theta\lambda_j^{i+1}(t_{i+1} - t_i) - \theta\eta_j^{i+1}\varepsilon \; \geq \; \hat{N}_j^{i+}(\theta\lambda_j^{i+1}(t_{i+1}^- - t_i^+))\Big\} \\
&\qquad\qquad \bigcup \Big\{ \hat{N}_j^{i+}(\theta\lambda_j^{i+1}(t_{i+1}^- - t_i^+)) \; \geq \; \theta\lambda_j^{i+1}(t_{i+1} - t_i) - \theta\zeta_j^i\varepsilon\Big\}\Big)\Big]
\end{aligned}$$

$$
\begin{aligned}
&\geq\ 1-\sum_{j\in S_{i+1}} \mathsf{P}\left[\hat{N}_j^{i+}(\theta\lambda_j^{i+1}(t_{i+1}^- - t_i^+))\ \leq\ \theta\lambda_j^{i+1}(t_{i+1}-t_i)-\theta\eta_j^{i+1}\varepsilon\right]\\
&\qquad\qquad -\sum_{j\in S_{i+1}} \mathsf{P}\left[\hat{N}_j^{i+}(\theta\lambda_j^{i+1}(t_{i+1}^- - t_i^+))\ \geq\ \theta\lambda_j^{i+1}(t_{i+1}-t_i)-\theta\zeta_j^{i}\varepsilon\right]\\
&=\ 1-\sum_{j\in S_{i+1}} \mathsf{P}\left[\hat{N}_j^{i+}(\theta\lambda_j^{i+1}(t_{i+1}^- - t_i^+))\ \leq\ \theta\lambda_j^{i+1}(t_{i+1}^- - t_i^+)+\theta\lambda_j^{i+1}(\nu_{i+1}^- + \nu_i^+)\varepsilon-\theta\eta_j^{i+1}\varepsilon\right]\\
&\qquad\qquad -\sum_{j\in S_{i+1}} \mathsf{P}\left[\hat{N}_j^{i+}(\theta\lambda_j^{i+1}(t_{i+1}^- - t_i^+))\ \geq\ \theta\lambda_j^{i+1}(t_{i+1}^- - t_i^+)+\theta\lambda_j^{i+1}(\nu_{i+1}^- + \nu_i^+)\varepsilon-\theta\zeta_j^{i}\varepsilon\right]\\
&\geq\ 1-\sum_{j\in S_{i+1}} \Big\{ \mathsf{P}\left[\hat{N}_j^{i+}(\theta\lambda_j^{i+1}(t_{i+1}^- - t_i^+))-\theta\lambda_j^{i+1}(t_{i+1}^- - t_i^+)\ \leq\ -\theta\lambda_j^{i+1}\varepsilon\right]\\
&\qquad\qquad +\mathsf{P}\left[\hat{N}_j^{i+}(\theta\lambda_j^{i+1}(t_{i+1}^- - t_i^+))-\theta\lambda_j^{i+1}(t_{i+1}^- - t_i^+)\ \geq\ \theta\lambda_j^{i+1}\varepsilon\right]\Big\}\\
&\geq\ 1-2\sum_{j\in S_{i+1}} \exp\left(-\theta\lambda_j^{i+1}(t_{i+1}^- - t_i^+)h\left(\frac{\varepsilon}{t_{i+1}^- - t_i^+}\right)\right)\ =\ 1-\delta_{i+1}^{\theta-}.
\end{aligned}
$$

The first inequality follows from Lemma A.15, the third equality holds because $\hat{N}_j^{i+}$ and $F_i^{\theta+}$ are independent, the second inequality applies the union bound, the third inequality follows from (A.6), and the fourth inequality follows from Lemma A.17.

Next we consider the conditional probability of $F_i^{\theta+}$ given $F_i^{\theta-}$.

$$
\begin{aligned}
&\mathsf{P}\left[F_i^{\theta+}\mid F_i^{\theta-}\right]\ =\ \mathsf{P}\left[E_i^{\theta+}\mid F_i^{\theta-}\right]\\
&=\ \mathsf{P}\left[\bigcap_{j\in S_i}\left\{\sum_{i'=1}^{i}\theta\lambda_j^{i'}(t_{i'}-t_{i'-1})\ \leq\ N_j^{\theta}(t_i^+)\ \leq\ \sum_{i'=1}^{i}\theta\lambda_j^{i'}(t_{i'}-t_{i'-1})+\theta\zeta_j^i\varepsilon\right\}\mid F_i^{\theta-}\right]\\
&=\ \mathsf{P}\Bigg[\bigcap_{j\in S_i\setminus S_{i+1}}\left\{\sum_{i'=1}^{i}\theta\lambda_j^{i'}(t_{i'}-t_{i'-1})-N_j^{\theta}(t_i^-)\ =\ N_j^{\theta}(t_i^+)-N_j^{\theta}(t_i^-)\right\}\\
&\qquad\qquad\bigcap_{j\in S_{i+1}}\Big\{\sum_{i'=1}^{i}\theta\lambda_j^{i'}(t_{i'}-t_{i'-1})-N_j^{\theta}(t_i^-)\ \leq\ N_j^{\theta}(t_i^+)-N_j^{\theta}(t_i^-)\\
&\qquad\qquad\qquad\qquad\leq\ \sum_{i'=1}^{i}\theta\lambda_j^{i'}(t_{i'}-t_{i'-1})-N_j^{\theta}(t_i^-)+\theta\zeta_j^i\varepsilon\Big\}\mid F_i^{\theta-}\Bigg]\\
&\geq\ \mathsf{P}\Bigg[\bigcap_{j\in S_i\setminus S_{i+1}}\left\{\sum_{i'=1}^{i}\theta\lambda_j^{i'}(t_{i'}-t_{i'-1})-N_j^{\theta}(t_i^-)\ =\ N_j^{\theta}(t_i^+)-N_j^{\theta}(t_i^-)\right\}\\
&\qquad\qquad\bigcap_{j\in S_{i+1}}\left\{\theta\eta_j^i\varepsilon\ \leq\ N_j^{\theta}(t_i^+)-N_j^{\theta}(t_i^-)\ \leq\ \theta\zeta_j^i\varepsilon\right\}\mid F_i^{\theta-}\Bigg]\\
&=\ \mathsf{P}\Bigg[\bigcap_{j\in S_i\setminus S_{i+1}}\left\{\sum_{i'=1}^{i}\theta\lambda_j^{i'}(t_{i'}-t_{i'-1})-N_j^{\theta}(t_i^-)\ =\ \hat{N}_j^{i-}\left(\theta\lambda\int_{\tau\in(t_i^-,t_i^+]}P_{j:S^{\theta}(\tau)}d\tau\right)\right\}
\end{aligned}
$$

$$
\begin{aligned}
&\bigcap_{j \in S_{i+1}} \left\{ \theta \eta_j^i \varepsilon \;\le\; \hat{N}_j^{i-} \left(\theta \lambda \int_{\tau \in (t_i^-, t_i^+]} P_{j:S^\theta(\tau)} d\tau \right) \;\le\; \theta \zeta_j^i \varepsilon \right\} \mid F_i^{\theta-} \Bigg] \\
\ge\;& \mathsf{P}\left[\bigcap_{j \in S_i} \left\{ \theta \eta_j^i \varepsilon \;\le\; \hat{N}_j^{i-} \left(\theta \underline{\lambda}_j^i (t_i^+ - t_i^-) \right) \right\} \bigcap_{j \in S_{i+1}} \left\{ \hat{N}_j^{i-} \left(\theta \overline{\lambda}_j^i (t_i^+ - t_i^-) \right) \;\le\; \theta \zeta_j^i \varepsilon \right\} \mid F_i^{\theta-} \right] \\
=\;& \mathsf{P}\left[\bigcap_{j \in S_i} \left\{ \theta \eta_j^i \varepsilon \;\le\; \hat{N}_j^{i-} \left(\theta \underline{\lambda}_j^i (t_i^+ - t_i^-) \right) \right\} \bigcap_{j \in S_{i+1}} \left\{ \hat{N}_j^{i-} \left(\theta \overline{\lambda}_j^i (t_i^+ - t_i^-) \right) \;\le\; \theta \zeta_j^i \varepsilon \right\} \right] \\
=\;& 1 - \mathsf{P}\left[\bigcup_{j \in S_i} \left\{ \theta \eta_j^i \varepsilon \;>\; \hat{N}_j^{i-} \left(\theta \underline{\lambda}_j^i (t_i^+ - t_i^-) \right) \right\} \bigcup_{j \in S_{i+1}} \left\{ \hat{N}_j^{i-} \left(\theta \overline{\lambda}_j^i (t_i^+ - t_i^-) \right) \;>\; \theta \zeta_j^i \varepsilon \right\} \right] \\
\ge\;& 1 - \sum_{j \in S_i} \mathsf{P}\left[\hat{N}_j^{i-} \left(\theta \underline{\lambda}_j^i (t_i^+ - t_i^-) \right) \;<\; \theta \eta_j^i \varepsilon \right] - \sum_{j \in S_{i+1}} \mathsf{P}\left[\hat{N}_j^{i-} \left(\theta \overline{\lambda}_j^i (t_i^+ - t_i^-) \right) \;>\; \theta \zeta_j^i \varepsilon \right] \\
\ge\;& 1 - \sum_{j \in S_i} \mathsf{P}\left[\hat{N}_j^{i-} \left(\theta \underline{\lambda}_j^i (t_i^+ - t_i^-) \right) \;<\; \theta \eta_j^i \varepsilon \right] - \sum_{j \in S_i} \mathsf{P}\left[\hat{N}_j^{i-} \left(\theta \overline{\lambda}_j^i (t_i^+ - t_i^-) \right) \;>\; \theta \zeta_j^i \varepsilon \right] \\
\ge\;& 1 - \sum_{j \in S_i} \mathsf{P}\left[\hat{N}_j^{i-} \left(\theta \underline{\lambda}_j^i (t_i^+ - t_i^-) \right) - \theta \underline{\lambda}_j^i (\nu_i^- + \nu_i^+) \varepsilon \;<\; -\theta \underline{\lambda}_j^i \varepsilon \right] \\
&\qquad - \sum_{j \in S_i} \mathsf{P}\left[\hat{N}_j^{i-} \left(\theta \overline{\lambda}_j^i (t_i^+ - t_i^-) \right) - \theta \overline{\lambda}_j^i (\nu_i^- + \nu_i^+) \varepsilon \;>\; \theta \overline{\lambda}_j^i \varepsilon \right] \\
=\;& 1 - \sum_{j \in S_i} \mathsf{P}\left[\hat{N}_j^{i-} \left(\theta \underline{\lambda}_j^i (t_i^+ - t_i^-) \right) - \theta \underline{\lambda}_j^i (t_i^+ - t_i^-) \;<\; -\theta \underline{\lambda}_j^i \varepsilon \right] \\
&\qquad - \sum_{j \in S_i} \mathsf{P}\left[\hat{N}_j^{i-} \left(\theta \overline{\lambda}_j^i (t_i^+ - t_i^-) \right) - \theta \overline{\lambda}_j^i (t_i^+ - t_i^-) \;>\; \theta \overline{\lambda}_j^i \varepsilon \right] \\
\ge\;& 1 - \sum_{j \in S_i} \exp\left(-\theta \underline{\lambda}_j^i (t_i^+ - t_i^-) h\left(\frac{\varepsilon}{t_i^+ - t_i^-} \right) \right) - \sum_{j \in S_i} \exp\left(-\theta \overline{\lambda}_j^i (t_i^+ - t_i^-) h\left(\frac{\varepsilon}{t_i^+ - t_i^-} \right) \right) \\
=\;& 1 - \delta_i^{\theta+}.
\end{aligned}
$$

The first inequality holds because $F_i^{\theta-} \subset E_i^{\theta-}$, and for all sample paths in $E_i^{\theta-}$ it holds that $\sum_{i'=1}^{i} \theta \lambda_j^{i'} (t_{i'} - t_{i'-1}) - \theta \eta_j^i \varepsilon < N_j^\theta(t_i^-) < \sum_{i'=1}^{i} \theta \lambda_j^{i'} (t_{i'} - t_{i'-1})$ for all $j \in S_i$, the second inequality follows from Lemma A.16, the fifth equality holds because $\hat{N}_j^{i-}$ and $F_i^{\theta-}$ are independent, the third inequality applies the union bound, the fourth inequality holds because $S_{i+1} \subset S_i$, the fifth inequality follows from (A.6), and the sixth inequality follows from Lemma A.17.

Therefore, using the induction hypothesis, it follows that

$$
\begin{aligned}
\mathsf{P}\left[F_{i+1}^{\theta+}\right] &= \mathsf{P}\left[F_{i+1}^{\theta+} \mid F_{i+1}^{\theta-}\right] \mathsf{P}\left[F_{i+1}^{\theta-} \mid F_{i}^{\theta+}\right] \mathsf{P}\left[F_{i}^{\theta+}\right] \\
&\geq \left(1-\delta_{i+1}^{\theta+}\right)\left(1-\delta_{i+1}^{\theta-}\right)\left(1-\sum_{i'=1}^{i}\left(\delta_{i'}^{\theta-}+\delta_{i'}^{\theta+}\right)\right) \geq 1-\sum_{i'=1}^{i+1}\left(\delta_{i'}^{\theta-}+\delta_{i'}^{\theta+}\right).
\end{aligned}
$$

Thus we have established (A.7). Since the expected sales quantity of any product during $[t_k, t_k^+]$ is bounded by $\theta\lambda(t_k^+ - t_k) = \theta O(\varepsilon)$, it follows that

$$
\begin{aligned}
\mathsf{E}\left[N_j^{\theta}(T)\right] &\geq \mathsf{E}\left[N_j^{\theta}(t_k^+)\right] - \theta O(\varepsilon) \\
&\geq \mathsf{E}\left[N_j^{\theta}(t_k^+) \mid F_k^{\theta+}\right] \mathsf{P}\left[F_k^{\theta+}\right] - \theta O(\varepsilon) \geq \theta b_j^* \left[1-\sum_{i=1}^{k}\left(\delta_i^{\theta-}+\delta_i^{\theta+}\right)\right] - \theta O(\varepsilon).
\end{aligned}
$$

For any fixed $\varepsilon > 0$, it holds that $\delta_i^{\theta-} \to 0$ and $\delta_i^{\theta+} \to 0$ for all i as $\theta \to \infty$. Thus

$$
\begin{aligned}
\liminf_{\theta\to\infty} \frac{1}{\theta} \mathsf{E}\left[Z^{\theta}\right] &= \liminf_{\theta\to\infty} \frac{1}{\theta} \sum_{j\in\mathcal{J}} r_j \, \mathsf{E}\left[N_j^{\theta}(T)\right] \\
&\geq \liminf_{\theta\to\infty} \sum_{j\in\mathcal{J}} r_j \left\{ b_j^* \left[1-\sum_{i=1}^{k}\left(\delta_i^{\theta-}+\delta_i^{\theta+}\right)\right] - O(\varepsilon) \right\} \\
&= \sum_{j\in\mathcal{J}} r_j b_j^* - O(\varepsilon) = z^{\mathrm{CDLP}} - O(\varepsilon).
\end{aligned}
$$

Since ε can be arbitrarily small, it follows that $\liminf_{\theta\to\infty} \frac{1}{\theta} \mathsf{E}\left[Z^{\theta}\right] \geq z^{\mathrm{CDLP}}$. Also, $\limsup_{\theta\to\infty} \frac{1}{\theta} \mathsf{E}\left[Z^{\theta}\right] \leq \limsup_{\theta\to\infty} \frac{1}{\theta} z_{OPT}^{\theta} \leq z^{\mathrm{CDLP}}$. Therefore $\lim_{\theta\to\infty} \frac{1}{\theta} \mathsf{E}\left[Z^{\theta}\right] = z^{\mathrm{CDLP}}$. $\square$

APPENDIX B
APPENDIX FOR CHAPTER 3

B.1 Proof of Theorem 3.3

The result below verifies the claim in Remark 3.2.

Claim B.1. *Suppose there are at least two loads with lead time ℓ (where $\ell \in \{1, \ldots, L\}$) under a given state $\boldsymbol{x}$, namely, $x_\ell \geq 2$. Then, under the MNL choice model, it is optimal to set the prices for these loads to be the same.*

Proof. Let $\mathscr{I}$ denote the set of loads that are currently in the system under state vector $\boldsymbol{x}$. For any load $i \in \mathscr{I}$, we denote its lead time by $L(i)$ and price by $p(i)$. Consider any two loads with lead time ℓ, which have indices i_1 and i_2 in set $\mathscr{I}$. Suppose a policy sets different prices for these loads such that $p(i_1) < p(i_2)$. We will prove that this policy is not optimal.

Under the MNL choice model defined in Section 3.4, the probability that any of these loads is booked in the current period is

$$\mathsf{P}\left[\text{Load 1 or load 2 is booked}\right] = \mu \frac{v_\ell(p(i_1)) + v_\ell(p(i_2))}{\sum_{i \in \mathscr{I}} v_{L(i)}(p(i)) + 1}.$$

Since function $v_\ell(p)$ is continuous, we can choose a new price $\bar{p} \in [p(i_1), p(i_2)]$ such that $2v_\ell(\bar{p}) = v_\ell(p(i_1)) + v_\ell(p(i_2))$. Let us change the prices of load i_1, i_2 both to $\bar{p}$, while keeping the prices of other loads. Then, the probability that load i_1 or i_2 is booked in the current period remains the same:

$$\begin{aligned}
&\mathsf{P}\left[\text{Load } i_1 \text{ or load } i_2 \text{ is booked}\right] \\
&= \mu \frac{v_\ell(\bar{p}) + v_\ell(\bar{p})}{\sum_{i \in \mathscr{I}/\{i_1, i_2\}} v_{L(i)}(p(i)) + 2v_\ell(\bar{p}) + 1} = \mu \frac{v_\ell(p(i_1)) + v_\ell(p(i_2))}{\sum_{i \in \mathscr{I}} v_{L(i)}(p(i)) + 1}.
\end{aligned}$$

The probability that any other load $j \in \mathscr{I}$ ($j \neq i_1, i_2$) is booked also remains the same:

$$\mathsf{P}\left[\text{Load } j \text{ is booked}\right] = \mu \frac{v_{L(j)}(p(j))}{\sum_{i \in \mathscr{I}/\{i_1,i_2\}} v_{L(i)}(p(i)) + 2v_\ell(\bar{p}) + 1} = \mu \frac{v_{L(j)}(p(j))}{\sum_{i \in \mathscr{I}} v_{L(i)}(p(i)) + 1}.$$

So changing prices of load i_1 and load i_2 to $\bar{p}$ does not affect the distribution of state variables in the next period. However, the expected payment to carriers in the current period is changed by

$$\mu \frac{2\bar{p}v_\ell(\bar{p})}{\sum_{i \in \mathscr{I}} v_{L(i)}(p(i)) + 1} - \mu \frac{p(i_1)v_\ell(p(i_1)) + p(i_2)v_\ell(p(i_2))}{\sum_{i \in \mathscr{I}} v_{L(i)}(p(i)) + 1} < 0.$$

The last inequality holds because $p_\ell(v) \cdot v$ is strictly convex in v, where $p_\ell(v)$ is the inverse function of $v_\ell(p) = \exp(\beta_0 + \beta_\ell p)$, and because of the definition of $v_\ell(\bar{p})$. $\square$

Then we establish the following monotonicity result for the optimal differential cost function h^*:

Lemma B.2. *In the Homogeneous Preference setting, for any state $\boldsymbol{x}$, and for any lead times $k < \ell$, it holds that $h^*(\boldsymbol{x} + \boldsymbol{e}^k) \geq h^*(\boldsymbol{x} + \boldsymbol{e}^\ell)$.*

Proof. We show a partial ordering on states: given two states $\boldsymbol{x}+\boldsymbol{e}^k$ and $\boldsymbol{x}+\boldsymbol{e}^\ell$ with $k < \ell$, the differential cost $h^*(\boldsymbol{x} + \boldsymbol{e}^k) \geq h^*(\boldsymbol{x} + \boldsymbol{e}^\ell)$. We note that $h^*(\boldsymbol{x})$ represents the average cost incurred as the system moves from state $\boldsymbol{x}$ to a reference state, e.g., $\boldsymbol{x}' = \boldsymbol{0}$, for the first time. To show our desired result $h^*(\boldsymbol{x} + \boldsymbol{e}^k) \geq h^*(\boldsymbol{x} + \boldsymbol{e}^\ell)$, it suffices to show that the following: initiating two chains with states $\boldsymbol{x} + \boldsymbol{e}^k$ and $\boldsymbol{x} + \boldsymbol{e}^\ell$ respectively, it is more costly for the first chain to reach a common reference state. We prove the result by coupling the two chains: 1) we apply the optimal price in each stage for the first chain; 2) we use the same price in the second chain for loads corresponding to $\boldsymbol{x}$ and apply the price for the load corresponding to $\boldsymbol{e}^k$ in the first chain (denoted by $\hat{k}$) to the load corresponding to $\boldsymbol{e}^\ell$ in the second chain (denoted by $\hat{\ell}$) until the stage that load $\hat{k}$ in the first chain expires or is chosen; 3) noting that the attraction of a load depends only on price but not on lead time in the Homogeneous Preference setting, we let the two chains be coupled such that they have the same new arrivals in each period and drivers make the same choice if they choose from $\boldsymbol{x}$ or choose load $\hat{k}$ from the first chain and load $\hat{\ell}$ from the second chain with the same probability; and 4) if load $\hat{k}$ in the first chain is chosen before it expires, load $\hat{\ell}$ in the second chain is also chosen; otherwise, recalling that $k < j$, we stop offering load $\hat{\ell}$ (e.g., by offering a very low price) in the second chain after load $\hat{k}$ expires and let $\hat{\ell}$ expire. From the above construction, we see that both chains reach a common state eventually and incur the same amount of cost. As we use the optimal price in each period for the first chain and a suboptimal price in each stage for the second chain, it follows that $h^*(\boldsymbol{x}+\boldsymbol{e}^k) \geq h^*(\boldsymbol{x}+\boldsymbol{e}^\ell)$. $\square$

Theorem 3.3. *In the Homogeneous Preference setting, in any state, the optimal price is higher for loads with a shorter lead time than for loads with a longer lead time. That is, for a given state $\boldsymbol{x}$, if $\boldsymbol{p}^*(\boldsymbol{x})$ is an optimal price vector, then $p_i^*(\boldsymbol{x}) \geq p_j^*(\boldsymbol{x})$ for any $i, j \in$ supp$(\boldsymbol{x})$ such that $i < j$.*

Proof. We show by contradiction that if there exist $i', j' \in \text{supp}(\boldsymbol{x})$ such that $i' < j'$ and $p_{i'}^*(\boldsymbol{x}) < p_{j'}^*(\boldsymbol{x})$ for some state $\boldsymbol{x}$, we can construct a better price that lowers the cost in state $\boldsymbol{x}$. We denote by $p_L = p_{i'}^*(\boldsymbol{x})$ and $p_H = p_{j'}^*(\boldsymbol{x})$.

Case 1: $x_{i'} = x_{j'}$.

We construct a new price, denoted by $\tilde{\boldsymbol{p}}(\boldsymbol{x})$, by perturbing the optimal policy with interchanging the prices on lead time i' and j'. That is, we set price $\tilde{p}_{i'}(\boldsymbol{x}) = p_H$ and $\tilde{p}_{j'}(\boldsymbol{x}) = p_L$, and $\tilde{p}_\ell(\boldsymbol{x}) = p_\ell^*(\boldsymbol{x})$ for $\ell \neq i', j'$ in the perturbed policy. In the Homogeneous Preference setting and noting that $x_i = x_j$, the above swap in price shuffles the choice probabilities of loads with lead time i' and j' but does not affect the choice probabilities of loads with lead times other than i' and j'.

Recalling the Bellman equation and letting

$$\overline{h}_\ell(\boldsymbol{x}) \;=\; \sum_{k=1}^{L} \lambda \psi_k h^* \left(\mathcal{S}\left(\boldsymbol{x} - \boldsymbol{e}^\ell\right) + \boldsymbol{e}^k\right) \;+\; (1-\lambda) h^* \left(\mathcal{S}\left(\boldsymbol{x} - \boldsymbol{e}^\ell\right)\right),$$

we can prove that the perturbed price is better by showing that

$$\begin{aligned} &f_{i'}(\boldsymbol{p}^*(\boldsymbol{x}),\boldsymbol{x})\left(p_L+\overline{h}_{i'}(\boldsymbol{x})\right)+f_{j'}(\boldsymbol{p}^*(\boldsymbol{x}),\boldsymbol{x})\left(p_H+\overline{h}_{j'}(\boldsymbol{x})\right) \\ &\qquad\qquad\geq\; f_{i'}(\tilde{\boldsymbol{p}}(\boldsymbol{x}),\boldsymbol{x})\left(p_H+\overline{h}_{i'}(\boldsymbol{x})\right)+f_{j'}(\tilde{\boldsymbol{p}}(\boldsymbol{x}),\boldsymbol{x})\left(p_L+\overline{h}_{j'}(\boldsymbol{x})\right) \end{aligned}$$

which, after plugging the multinomial logit model and arranging terms, is equivalent to

$$v(p_H)(\overline{h}_{j'}(\boldsymbol{x})-\overline{h}_{i'}(\boldsymbol{x}))\;\geq\; v(p_L)(\overline{h}_{j'}(\boldsymbol{x})-\overline{h}_{i'}(\boldsymbol{x})).$$

The last inequality holds as $v(p_H) > v(p_L)$ and $\overline{h}_{j'}(\boldsymbol{x}) \geq \overline{h}_{i'}(\boldsymbol{x})$ by our argument in the first part of the proof. This contradicts the optimality assumption on the price $\boldsymbol{p}^*(\boldsymbol{x})$.

Case 2: $x_{i'} > x_{j'}$.

We show that we can also construct a better price in this case. First, for $x_{j'}$ out of $x_{i'}$ of the loads with lead time i', we swap their price with the price of the $x_{j'}$ loads with lead time j'. Then we set a price p_M for the $x_{i'}$ loads with lead time i' such that

$$x_{j'}v(p_H)+(x_{i'}-x_{j'})v(p_L)=x_{i'}v(p_M). \tag{B.1}$$

Then the new perturbed price $\tilde{p}(\boldsymbol{x})$ is given by $\tilde{p}_{i'}(\boldsymbol{x}) = p_M$ and $\tilde{p}_{j'}(\boldsymbol{x}) = p_L$, and $\tilde{p}_\ell(\boldsymbol{x}) = p^*_\ell(\boldsymbol{x})$ for $\ell \neq i', j'$. We note that the choice probabilities of loads with lead times other than i' and j', as well as the total choice probability, are not affected.

Similar to the previous case, we aim to show that

$$\begin{aligned} &f_{i'}(\boldsymbol{p}^*(\boldsymbol{x}),\boldsymbol{x})\left(p_L+\overline{h}_{i'}(\boldsymbol{x})\right)+f_{j'}(\boldsymbol{p}^*(\boldsymbol{x}),\boldsymbol{x})\left(p_H+\overline{h}_{j'}(\boldsymbol{x})\right) \\ &\qquad\qquad\geq\; f_{i'}(\tilde{\boldsymbol{p}}(\boldsymbol{x}),\boldsymbol{x})\left(p_H+\overline{h}_{i'}(\boldsymbol{x})\right)+f_{j'}(\tilde{\boldsymbol{p}}(\boldsymbol{x}),\boldsymbol{x})\left(p_L+\overline{h}_{j'}(\boldsymbol{x})\right), \end{aligned}$$

which is equivalent to

$$\begin{aligned}
& x_{i'} v(p_L)\left(p_L + \overline{h}_{i'}(\boldsymbol{x})\right) + x_{j'} v(p_H)\left(p_H + \overline{h}_{j'}(\boldsymbol{x})\right) \\
\geq\ & x_{i'} v(p_M)\left(p_M + \overline{h}_{i'}(\boldsymbol{x})\right) + x_{j'} v(p_L)\left(p_L + \overline{h}_{j'}(\boldsymbol{x})\right) \\
=\ & \left(x_{j'} v(p_H) + (x_{i'} - x_{j'}) v(p_L)\right)\left(p_M + \overline{h}_{i'}(\boldsymbol{x})\right) + x_{j'} v(p_L)\left(p_L + \overline{h}_{j'}(\boldsymbol{x})\right).
\end{aligned}$$

Arranging terms in the above inequality yields

$$\begin{aligned}
& x_{j'} v(p_H)(p_H - p_M) + x_{j'} v(p_H)(\overline{h}_{j'}(\boldsymbol{x}) - \overline{h}_{i'}(\boldsymbol{x})) \\
& \qquad \geq\ (x_{i'} - x_{j'}) v(p_L)(p_M - p_L) + x_{j'} v(p_L)(\overline{h}_{j'}(\boldsymbol{x}) - \overline{h}_{i'}(\boldsymbol{x})).
\end{aligned}$$

Since $x_{j'} v(p_H)(\overline{h}_{j'}(\boldsymbol{x}) - \overline{h}_{i'}(\boldsymbol{x})) \ \geq\ x_{j'} v(p_L)(\overline{h}_{j'}(\boldsymbol{x}) - \overline{h}_{i'}(\boldsymbol{x}))$, it suffices to show that

$$\begin{aligned}
& x_{j'} v(p_H)(p_H - p_M) \ \geq\ (x_{i'} - x_{j'}) v(p_L)(p_M - p_L) \\
\iff\ & x_{j'} v(p_H) p_H + (x_{i'} - x_{j'}) v(p_L) p_L \ \geq\ x_{j'} v(p_H) p_M + (x_{i'} - x_{j'}) v(p_L) p_M \\
\overset{(a)}{\iff}\ & x_{j'} v(p_H) p_H + (x_{i'} - x_{j'}) v(p_L) p_L \ \geq\ x_{i'} v(p_M) p_M,
\end{aligned}$$

where (a) follows from (B.1). Letting $\alpha = x_{j'}/x_{i'}$ and $u_k = v(p_k)$ for $k \in \{H, M, L\}$, we derive from (B.1) that

$$\alpha u_H + (1 - \alpha) u_L = u_M.$$

Noting that $p_k = \frac{1}{\beta}(\ln(u_k) - \beta_0)$ for $k \in \{H, M, L\}$ and that the function $\tilde{r}(u) = \frac{u}{\beta}(\ln(u) - \beta_0)$ is convex, we conclude by Jensen's inequality that

$$\alpha \tilde{r}(u_H) + (1 - \alpha)\tilde{r}(u_L) \ \geq\ \tilde{r}(u_M),$$

which is the result $x_{j'} v(p_H) p_H + (x_{i'} - x_{j'}) v(p_L) p_L \ \geq\ x_{i'} v(p_M) p_M$ that we want. Therefore, we also have a contradiction in this case.

Case 3: $x_{i'} < x_{j'}$. Using similar reasoning as in the previous case, we have the same conclusion.

Putting things together, we conclude that the optimal price $p^*(\boldsymbol{x})$ in any state $\boldsymbol{x}$ must satisfy that $p_i^*(\boldsymbol{x}) \geq p_j^*(\boldsymbol{x})$ for any $i, j \in \text{supp}(\boldsymbol{x})$ such that $i < j$. $\square$

B.2 Proof of Theorem 3.4

Recall that we let $\boldsymbol{\varphi} : \mathbb{N}^{L+1} \mapsto \mathbb{R}_+^L$ denote a stationary pricing policy for the MDP. There exists a unique stationary distribution of the system state $\boldsymbol{X}$ under any stationary policy $\boldsymbol{\varphi}$. Then, for any stationary policy $\boldsymbol{\varphi}$, the average cost per discrete period for the platform under the multinomial logit model is given by

$$R(\boldsymbol{\varphi}) \ := \ \mathsf{E}\left[\sum_{\ell=1}^{L} \mu \frac{\varphi_\ell(\boldsymbol{X}) \ X_\ell \ v_\ell(\varphi_\ell(\boldsymbol{X}))}{\sum_{k=1}^{L} X_k \ v_k(\varphi_k(\boldsymbol{X})) + 1} + C \ X_0\right],$$

where the expectation is taken with respect to the stationary distribution of $\boldsymbol{X}$.

We consider the Convex DTFM, which is equivalent to the DTFM.

$$\hat{R} \ := \min_{(\boldsymbol{x},\boldsymbol{u},u_0)\in\mathbb{R}_+^{2(L+1)}} \quad \frac{\mu}{\beta}\sum_{\ell=1}^{L} u_\ell \left(\ln\left(\frac{u_\ell}{x_\ell}\right) - \beta_\ell^0\right) - \frac{\mu}{\beta}(1-u_0)\ln(u_0) + C \ x_0 \quad \text{(B.2a)}$$

$$\text{s.t.} \quad x_\ell - x_{\ell-1} \ = \ \mu u_\ell - \lambda\psi_{\ell-1} \qquad \text{for } \ell = 1,\dots,L, \quad \text{(B.2b)}$$

$$x_L \ = \ \lambda\psi_L \quad \text{(B.2c)}$$

$$\sum_{\ell=1}^{L} u_\ell + u_0 \ = \ 1 \quad \text{(B.2d)}$$

$$x_\ell \ \geq \ \mu u_\ell \qquad \text{for } \ell = 1,\dots,L. \quad \text{(B.2e)}$$

Theorem 3.4. *The optimal objective value of the discrete-time fluid model* (DTFM) *is a lower bound for the long-run average cost per period of the MDP under any stationary policy $\boldsymbol{\varphi}$, i.e., $\hat{R} \leq R(\boldsymbol{\varphi})$ for all $\boldsymbol{\varphi}$.*

Proof. Let $\boldsymbol{X}$ be a steady state of the stochastic system, which follows the stationary distribution under the stationary policy $\hat{\boldsymbol{\varphi}}$. Define function

$$Z(\boldsymbol{\varphi}(\boldsymbol{X}), \boldsymbol{X}) \ := \ \sum_{\ell=1}^{L} \mu \frac{\varphi_\ell(\boldsymbol{X}) \ X_\ell \ v_\ell(\varphi_\ell(\boldsymbol{X}))}{\sum_{k=1}^{L} X_k \ v_k(\varphi_k(\boldsymbol{X})) + 1} + C \ X_0.$$

We perform variable substitution and define the following two functions $\tilde{\boldsymbol{u}} : \mathbb{R} \times \{0,1\}^{L+1} \mapsto \mathbb{R}_+^L$ and $\tilde{u}_0 : \mathbb{R}^L \times \{0,1\}^{L+1} \mapsto \mathbb{R}_+$ given by

$$\tilde{u}_\ell(\boldsymbol{p}, \boldsymbol{x}) = \frac{x_\ell \ v_\ell(p_\ell)}{\sum_{k=1}^{L} x_k \ v_k(p_k) + 1}, \quad \text{for } \ell = 1, \ldots, L,$$

and

$$\tilde{u}_0(\boldsymbol{p}, \boldsymbol{x}) = \frac{1}{\sum_{k=1}^{L} x_k \ v_k(p_k) + 1}.$$

And, we let $U_\ell = \tilde{u}_\ell(\boldsymbol{\varphi}(\boldsymbol{X}), \boldsymbol{X})$ and $\boldsymbol{U} = (U_\ell, \ell = 1, \ldots, \theta)$, and $U_0 = \tilde{u}_0(\boldsymbol{\varphi}(\boldsymbol{X}), \boldsymbol{X}) > 0$.

Then we have

$$\sum_{\ell=1}^{L} U_\ell + U_0 \ = \ 1, \tag{B.3}$$

and

$$\frac{U_\ell}{U_0} \ = \ X_\ell \ v_\ell(\varphi_\ell(\boldsymbol{X})) \quad \Rightarrow \quad \varphi_\ell(\boldsymbol{X}) \ = \ \frac{1}{\beta}\left(\ln\left(\frac{U_\ell}{X_\ell U_0}\right) - \beta_\ell^0\right) \quad \text{if } X_\ell > 0. \tag{B.4}$$

Substituting $\boldsymbol{U}$ and U_0 into the expression of $Z(\boldsymbol{\varphi}(\boldsymbol{X}), \boldsymbol{X})$, we have

$$\begin{aligned} & Z(\boldsymbol{\varphi}(\boldsymbol{X}), \boldsymbol{X}) \\ \overset{(a)}{=} \ & \sum_{\ell=1}^{L} \mu U_\ell \ \frac{1}{\beta}\left(\ln\left(\frac{U_\ell}{X_\ell U_0}\right) - \beta_\ell^0\right) + C \ X_0^\theta \end{aligned}$$

$$
\begin{aligned}
&\overset{(b)}{=} \frac{\mu}{\beta} \sum_{\ell=1}^{L} U_\ell \left(\ln \left(\frac{U_\ell}{X_\ell} \right) - \beta_\ell^0 \right) - \frac{\mu}{\beta} (1 - U_0) \ln (U_0) + C\ X_0 \\
&\overset{(c)}{=} g(\boldsymbol{X}, \boldsymbol{U}, U_0),
\end{aligned}
$$

where equality (a) holds due to variable substitutions and (B.4); equality (b) utilizes the fact (B.3); and equality (c) follows from the definition of function g. Noting that $g(\boldsymbol{x}, \boldsymbol{u}, u_0)$ is convex in its arguments, by Jensen's inequality, we have the long-run average cost

$$
R(\boldsymbol{\varphi}) \;=\; \mathsf{E}\left[Z(\boldsymbol{\varphi}(\boldsymbol{X}), \boldsymbol{X})\right] \;=\; \mathsf{E}\left[g(\boldsymbol{X}, \boldsymbol{U}, U_0)\right] \;\geq\; g(\mathsf{E}[\boldsymbol{X}], \mathsf{E}[\boldsymbol{U}], \mathsf{E}[U_0]).
$$

We let $\bar{\boldsymbol{x}} = \mathsf{E}[\boldsymbol{X}]$, $\bar{\boldsymbol{u}} = \mathsf{E}[\boldsymbol{U}]$ and $\bar{u}_0 = \mathsf{E}[U_0]$. Next, we show that $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{u}}, \bar{u}_0)$ satisfies the constraints (B.2b)-(B.2e).

Given a current state $\boldsymbol{X}$, which follows the stationary distribution under policy $\boldsymbol{\varphi}$, we let $\tilde{\boldsymbol{X}}$ be the next system state and then we have

$$
\mathsf{E}[\tilde{X}_{\ell-1} \mid \boldsymbol{X}] \;=\; X_\ell - \mu \frac{X_\ell\, v_\ell(\varphi_\ell(\boldsymbol{X}))}{\sum_{k=1} X_k\, v_k(\varphi_k(\boldsymbol{X})) + 1} + \mathsf{E}[A_{\ell-1}] \;=\; X_\ell - \mu U_\ell + \lambda \psi_{\ell-1},
$$

where $\boldsymbol{A} = (A_\ell)$ represents the random arrival of new loads, which is independent of $\boldsymbol{X}$. Since $\tilde{\boldsymbol{X}}$ also follows the stationary distribution by the property of stationary Markov chain. Taking expectation with respect to the stationary distribution under policy $\boldsymbol{\varphi}$ on both sides and using the fact that $\bar{\boldsymbol{x}} = \mathsf{E}[\boldsymbol{X}] = \mathsf{E}[\tilde{\boldsymbol{X}}]$, we see that $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{u}}, \bar{u}_0)$ satisfies constraint (B.2b). Constraint (B.2c) holds trivially. And, constraint (B.2d) follows from (B.3). Let $\boldsymbol{M} = (M_\ell)$ represents the random bookings of loads given state $\boldsymbol{X}$. Then, constraint (B.2e) holds since

$$
\begin{aligned}
& X_\ell \;\geq\; M_\ell \quad \Rightarrow \quad \mathsf{E}\, X_\ell \;\geq\; \mathsf{E}\left[\mathsf{E}\left[M_\ell \mid \boldsymbol{X}\right]\right] \\
\Rightarrow \quad & \mathsf{E}\, X_\ell \;\geq\; \mu\, \mathsf{E}\left[\frac{X_\ell\, v_\ell(\varphi_\ell(\boldsymbol{X}))}{\sum_{k=1} X_k\, v_k(\varphi_k(\boldsymbol{X})) + 1}\right] \;=\; \mu\, \mathsf{E}\, U_\ell.
\end{aligned}
$$

Therefore, the tuple $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{u}}, \bar{u}_0)$ is a feasible solution for problem (B.2). Since solving problem (B.2) minimizes the objective, we conclude that $R(\boldsymbol{\varphi}) \geq \hat{R}$ for any stationary policy $\boldsymbol{\varphi}$. $\square$

B.3 Proof of Lemma 3.5

Lemma 3.5. *An optimal solution $(\hat{\boldsymbol{x}}^\theta, \hat{\boldsymbol{p}}^\theta)$ for the θ-scaled DTFM satisfies $\hat{x}^\theta_\ell \leq \lambda$ for $\ell = 0, \ldots, \theta$ and $\hat{p}^\theta_\ell \leq C$ for $\ell = 1, \ldots, \theta$. Also, preference weight $\hat{v}^\theta_\ell := v^\theta_\ell(\hat{p}^\theta_\ell) \leq K_v := \exp(\beta C + b_{\max})$ for all $\ell = 1, \ldots, \theta$ and all θ.*

Proof. First, we give a convex reformulation of the θ-scaled DTFM. Let $(\boldsymbol{x}^\theta, \boldsymbol{u}^\theta, u^\theta_0) \in \mathbb{R}_+^{2(\theta+1)}$ be the decision variables. Let

$$u^\theta_0 = \frac{1}{\sum_{k=1}^\theta x^\theta_k \, v^\theta_k(p^\theta_k)/\theta \, + \, 1}, \quad u^\theta_\ell = \frac{x^\theta_\ell \, v^\theta_\ell(p^\theta_\ell)}{\sum_{k=1}^\theta x^\theta_k \, v^\theta_k(p^\theta_k)/\theta \, + \, 1}, \quad \forall \ell \geq 1.$$

The corresponding reformulation is given by

$$\min_{(\boldsymbol{x}^\theta, \boldsymbol{u}^\theta, u^\theta_0) \in \mathbb{R}_+^{2(\theta+1)}} \quad \frac{\mu}{\beta\theta} \sum_{\ell=1}^\theta u^\theta_\ell \left(\ln\left(\frac{u^\theta_\ell}{x^\theta_\ell} \right) - \beta^\theta_{0,\ell} \right) \; - \; \frac{\mu}{\beta}(1 - u^\theta_0)\ln(u^\theta_0) \; + \; C\, x^\theta_0 \tag{B.5a}$$

$$\text{s.t.} \quad x^\theta_\ell - x^\theta_{\ell-1} \; = \; \mu u^\theta_\ell/\theta - \lambda\psi^\theta_{\ell-1} \qquad \ell = 1, \ldots, \theta \tag{B.5b}$$

$$x^\theta_\theta \; = \; \lambda\psi^\theta_\theta \tag{B.5c}$$

$$\sum_{\ell=1}^\theta u^\theta_\ell/\theta + u^\theta_0 \; = \; 1. \tag{B.5d}$$

$$x^\theta_\ell \; \geq \; \mu u^\theta_\ell/\theta \qquad \ell = 1, \ldots, \theta \tag{B.5e}$$

The θ-scaled DTFM problem and problem (B.5) are equivalent, in that we can derive a feasible solution for one problem from a feasible solution for the other, with the two problems having the same objective value.

Let $(\hat{\boldsymbol{x}}^\theta, \hat{\boldsymbol{u}}^\theta, \hat{u}^\theta_0)$ be an optimal solution for the above problem and note that $\hat{\boldsymbol{x}}^\theta$ is also optimal for the θ-scaled DTFM. We first bound $\hat{\boldsymbol{x}}^\theta$. Noting that $\hat{x}^\theta_\theta = \lambda\psi^\theta_\theta \leq \lambda$, it follows by induction on ℓ that

$$
\begin{aligned}
\hat{x}_\ell^\theta &= \hat{x}_{\ell+1}^\theta + \lambda\psi_\ell^\theta - \mu\hat{u}_{\ell+1}^\theta/\theta \\
&= \cdots = \sum_{k=\ell}^{\theta}\lambda\psi_k^\theta - \sum_{k=\ell+1}^{\theta}\mu\hat{u}_k^\theta/\theta \le \sum_{k=\ell}^{\theta}\lambda\psi_k^\theta \le \lambda \qquad \text{for } \ell = 0,\dots,\theta-1.
\end{aligned}
$$

Next, we bound $\hat{\boldsymbol{p}}^\theta$.

Part 1: Relating Optimal Prices to Dual Variables

Let $\boldsymbol{\pi}^\theta := (\pi_\ell^\theta, \ell = 1,\dots,\theta)$, η^θ, ζ^θ and $\boldsymbol{\nu}^\theta := (\nu_\ell^\theta, \ell = 1,\dots,\theta) \in \mathbb{R}_+^\theta$ be the dual variables corresponding to constraints (B.5b)-(B.5e), respectively. Note that the objective function implies $(\boldsymbol{x}^\theta, \boldsymbol{u}^\theta, u_0^\theta) > \mathbf{0}$, we consider the Lagrangian function given by

$$
\begin{aligned}
&\mathcal{L}^\theta(\boldsymbol{x}^\theta, \boldsymbol{u}^\theta, u_0^\theta, \boldsymbol{\pi}^\theta, \eta^\theta, \zeta^\theta, \boldsymbol{\nu}^\theta) \\
&\quad = C\, x_0^\theta + \frac{\mu}{\beta\theta}\sum_{\ell=1}^{\theta} u_\ell^\theta\left(\ln\left(\frac{u_\ell^\theta}{x_\ell^\theta}\right) - \beta\left(-\pi_\ell^\theta + \frac{\zeta^\theta}{\mu} - \nu_\ell^\theta\right) - \beta_{0,\ell}^\theta\right) \\
&\qquad - \frac{\mu}{\beta}(1 - u_0^\theta)\left(\ln(u_0^\theta) - \frac{\beta\zeta^\theta}{\mu}\right) - \sum_{\ell=1}^{\theta}\pi_\ell^\theta(x_\ell^\theta - x_{\ell-1}^\theta + \lambda\psi_{\ell-1}^\theta) \\
&\qquad - \eta^\theta(x_\theta^\theta - \lambda\psi_\theta^\theta) - \sum_{\ell=1}^{\theta}\nu_\ell^\theta x_\ell^\theta.
\end{aligned}
$$

Let $(\hat{\boldsymbol{\pi}}^\theta, \hat{\eta}^\theta, \hat{\zeta}^\theta, \hat{\boldsymbol{\nu}}^\theta)$ be optimal dual variables. Since problem (B.5) is convex, the optimal solution satisfies the first order condition. We have

$$
\begin{aligned}
\frac{\partial\, \mathcal{L}^\theta(\hat{\boldsymbol{x}}^\theta, \hat{\boldsymbol{u}}^\theta, \hat{u}_0^\theta, \hat{\boldsymbol{\pi}}^\theta, \hat{\eta}^\theta, \hat{\zeta}^\theta)}{\partial\, u_\ell^\theta} &= \frac{\mu}{\beta\theta}\left(\ln\left(\frac{\hat{u}_\ell^\theta}{\hat{x}_\ell^\theta}\right) - \beta\left(-\hat{\pi}_\ell^\theta + \frac{\hat{\zeta}^\theta}{\mu} - \nu_\ell^\theta\right) - \beta_{0,\ell}^\theta + 1\right) = 0 \\
\frac{\partial\, \mathcal{L}^\theta(\hat{\boldsymbol{x}}^\theta, \hat{\boldsymbol{u}}^\theta, \hat{u}_0^\theta, \hat{\boldsymbol{\pi}}^\theta, \hat{\eta}^\theta, \hat{\zeta}^\theta)}{\partial\, u_0^\theta} &= \frac{\mu}{\beta}\left(\ln(\hat{u}_0^\theta) - \frac{\beta\hat{\zeta}^\theta}{\mu} + \frac{\hat{u}_0^\theta - 1}{\hat{u}_0^\theta}\right) = 0.
\end{aligned}
$$

Eliminating $\hat{\zeta}^\theta$ and arranging terms in the above equations, we have

$$\hat{u}_\ell^\theta \;= \hat{x}_\ell^\theta \exp\left(-\beta(\hat{\pi}_\ell^\theta + \nu_\ell^\theta) + \ln(\hat{u}_0^\theta) + \frac{\hat{u}_0^\theta - 1}{\hat{u}_0^\theta} + \beta_{0,\ell}^\theta - 1\right) \qquad \text{for } \ell = 1, \ldots, \theta.$$

Then the optimal prices for the θ-scaled DTFM are given by

$$\hat{p}_\ell^\theta \;= \frac{1}{\beta}\left(\ln\left(\frac{\hat{u}_\ell^\theta}{\hat{x}_\ell^\theta \hat{u}_0^\theta}\right) - \beta_{0,\ell}^\theta\right) \;=\; -\,\hat{\pi}_\ell^\theta - \nu_\ell^\theta - \frac{1}{\beta \hat{u}_0^\theta} \;\leq\; -\,\hat{\pi}_\ell^\theta \qquad \text{for } \ell = 1, \ldots, \theta.$$

Part 2: Bounding Dual Variables

We first introduce auxiliary variables $\boldsymbol{t}^\theta = (t_\ell^\theta, \ell = 1, \ldots, \theta)$ and s. We introduce the following constraints

$$-t_\ell^\theta \;\geq u_\ell^\theta \ln\left(\frac{u_\ell^\theta}{x_\ell^\theta}\right) \quad \text{for } \ell = 1, \ldots, \theta, \quad \text{and} \quad s \;\geq\; (1 - u_0^\theta) \ln\left(1 + \frac{1 - u_0^\theta}{u_0^\theta}\right),$$

which are conic representable. Replacing the corresponding nonlinear parts in the objective (B.5a) with $-\boldsymbol{t}^\theta$ and s and noting that as we are minimizing the objective, the introduced constraints are binding at the optimum. So, problem (B.5) is equivalent to the following conic program:

$$\begin{aligned}
\min \quad & \frac{\mu}{\beta\theta} \sum_{\ell=1}^{\theta} \left(-t_\ell^\theta - \beta_{0,\ell}^\theta u_\ell^\theta\right) \;+\; \frac{\mu}{\beta}\, s \;+\; C\, x_0^\theta \\
\text{s.t.} \quad & \text{(B.5b), (B.5c), (B.5d), (B.5e)}, \\
& (x_\ell^\theta, u_\ell^\theta, t_\ell^\theta) \in \mathcal{K}_{\exp} \qquad \text{for } \ell = 1, \ldots, \theta, \\
& (v, u_0^\theta, w_1) \in \mathcal{K}_{\exp}, \quad (u_0^\theta, v, w_2) \in \mathcal{K}_{\exp}, \quad s + w_1 + w_2 = 0, \quad v = 1, \\
& x_0^\theta \;\geq\; 0,
\end{aligned}$$

where $\mathcal{K}_{\exp} \;=\; \mathbf{cl}\,\{(x_1, x_2, x_3) \;:\; x_1 \geq x_2 \exp(x_3/x_2),\; x_2 > 0\}$ is the exponential cone. We note that the signs of the variables other than x_0^θ are implied by the conic constraints.

Let $(\boldsymbol{\pi}^\theta, \eta^\theta, \zeta^\theta, \boldsymbol{\nu}^\theta)$ be dual variables to the linear constraints and $(y^\theta_{\ell,1}, y^\theta_{\ell,2}, y^\theta_{\ell,3})$ with $\ell = 1, \ldots, \theta$ be the corresponding dual variables for the first θ conic constraints, and let $(z^\theta_{i,1}, z^\theta_{i,2}, z^\theta_{i,3})$ for $i \in \{1,2\}$ be the corresponding dual variables for the last two conic constraints. Let σ^θ be the dual variable to constraint $v = 1$. The dual problem to the above conic program, after simplification, is

$$\max \quad \sum_{\ell=1}^{\theta} -\lambda\psi^\theta_{\ell-1}\pi^\theta_\ell \;+\; \lambda\psi^\theta_\theta\eta^\theta \;+\; \zeta^\theta \;+\; \sigma^\theta \tag{B.6a}$$

$$\text{s.t.} \quad -\pi^\theta_1 \;\le\; C \tag{B.6b}$$

$$\begin{cases} y^\theta_{\ell,1} \;=\; \pi^\theta_{\ell+1} - \pi^\theta_\ell - \nu^\theta_\ell \\ y^\theta_{\ell,2} \;=\; -\frac{\mu\beta^\theta_{0,\ell}}{\beta\theta} + \frac{\mu(\pi^\theta_\ell + \nu^\theta_\ell)}{\theta} - \frac{\zeta^\theta}{\theta} \\ y^\theta_{\ell,3} \;=\; -\frac{\mu}{\beta\theta} \\ (y^\theta_{\ell,1}, y^\theta_{\ell,2}, y^\theta_{\ell,3}) \in \mathcal{K}^*_{\exp} \end{cases} \quad \text{for } \ell = 1, \ldots, \theta - 1 \tag{B.6c}$$

$$\begin{cases} y^\theta_{\theta,1} \;=\; -\pi^\theta_\theta - \eta^\theta - \nu^\theta_\theta \\ y^\theta_{\theta,2} \;=\; -\frac{\mu\beta^\theta_{0,\theta}}{\beta\theta} + \frac{\mu(\pi^\theta_\theta + \nu^\theta_\ell)}{\theta} - \frac{\zeta^\theta}{\theta} \\ y^\theta_{\theta,3} \;=\; -\frac{\mu}{\beta\theta} \\ (y^\theta_{\theta,1}, y^\theta_{\theta,2}, y^\theta_{\theta,3}) \in \mathcal{K}^*_{\exp} \end{cases} \tag{B.6d}$$

$$\begin{cases} z^\theta_{1,1} \;+\; z^\theta_{2,2} \;=\; -\sigma^\theta \\ z^\theta_{1,2} \;+\; z^\theta_{2,1} \;=\; -\zeta^\theta \\ z^\theta_{1,3} \;=\; z^\theta_{2,3} \;=\; -\frac{\mu}{\beta} \\ (z^\theta_{1,1}, z^\theta_{1,2}, z^\theta_{1,3}) \in \mathcal{K}^*_{\exp}, \quad (z^\theta_{2,1}, z^\theta_{2,2}, z^\theta_{2,3}) \in \mathcal{K}^*_{\exp} \end{cases} \tag{B.6e}$$

$$\boldsymbol{\nu}^\theta \ge \mathbf{0}$$

where $\mathcal{K}_{\exp}^* = \mathbf{cl}\{(y_1, y_2, y_3) : y_1 \geq -y_3 \exp(y_2/y_3 - 1),\ y_1 > 0,\ y_3 < 0\}$ is the dual cone of $\mathcal{K}_{\exp}$.

Either the primal conic program or its dual problem (B.6) is strictly feasible: There exists a point in the relative interior of the feasible region of the problem. Then strong duality holds. Let $\hat{\boldsymbol{\pi}}^\theta = (\hat{\pi}_i^\theta)$ denote the optimal dual variables to (B.5b) and $\hat{\boldsymbol{\nu}}^\theta = (\hat{\nu}_i^\theta) \in \mathbb{R}_+^\theta$ denote the optimal dual variables to (B.5e), which satisfies (B.6b)-(B.6d). Noting that $-\hat{\pi}_1^\theta \leq C$ by (B.6b), then by recursively examining (B.6c), we have

$$\begin{aligned} & \hat{y}_{\ell-1,1}^\theta \ = \ \hat{\pi}_\ell^\theta - \hat{\pi}_{\ell-1}^\theta - \hat{\nu}_{\ell-1}^\theta \ \geq \ 0 \\ \Rightarrow \quad & -\hat{\pi}_\ell^\theta \ \leq \ -\hat{\pi}_\ell^\theta + \hat{\nu}_{\ell-1}^\theta \ \leq \ -\hat{\pi}_{\ell-1}^\theta \ \leq \ C \qquad \text{for } \ell = 2, \ldots, \theta. \end{aligned}$$

Conclusion

Putting the results in the above two parts together, we conclude that $\hat{p}_\ell^\theta \leq C$ for all ℓ. Then, correspondingly, preference weight

$$\hat{v}_\ell^\theta \ = \ v_\ell^\theta(\hat{p}_\ell^\theta) \ = \ \exp(\beta \hat{p}_\ell^\theta + \beta_{0,\ell}^\theta) \ \leq \ \exp(\beta C + b_{\max}) \ =: \ K_v$$

for $\ell = 1, \ldots, \theta$ and all θ, where $b_{\max}$ is an upper bound of function b_0. □

B.4 Proofs of Lemma 3.9, Lemma 3.10, and Lemma 3.11

We recall and define notations that will be used throughout the proof. Recall that $(\hat{\boldsymbol{x}}^\theta, \hat{\boldsymbol{p}}^\theta)$ denote an optimal solution of the θ-scaled DTFM. The open-loop pricing policy $\hat{\boldsymbol{\varphi}}^\theta$ sets price $\hat{p}_\ell^\theta$ for all loads with lead time ℓ, i.e., $\hat{\varphi}_\ell^\theta(\boldsymbol{x}^\theta) = \hat{p}_\ell^\theta$ for all $\boldsymbol{x}^\theta$ and ℓ. As prices are fixed, we let preference weight $\hat{v}_\ell^\theta := v_\ell^\theta(\hat{p}_\ell^\theta)$ for $\ell = 1, \ldots, \theta$. Let $\boldsymbol{X}^\theta$ denote the state of the θ-scaled system, which follows the stationary distribution under the open-loop pricing policy $\hat{\boldsymbol{\varphi}}^\theta$, and let $\bar{\boldsymbol{x}}^\theta = \mathsf{E}[\boldsymbol{X}^\theta]$.

Also, recall Assumption 3.6: There exists a positive constant $\underline{p} > 0$ such that for all

$\theta \in \mathbb{N}$ and $\ell = 1, \ldots, \theta$, we have $\hat{p}^\theta \geq \underline{p}$.

B.4.1 Auxiliary Lemma B.3

Lemma B.3. *Under the steady state distribution of the open-loop pricing policy, the mean and variance of the state variable $\boldsymbol{X}^\theta$ is bounded by $\bar{x}^\theta_\ell = \mathsf{E}\left[X^\theta_\ell\right] \leq 1$ and $\mathsf{Var}(X^\theta_\ell) \leq \mathsf{E}\left[(X^\theta_\ell)^2\right] \leq 2$ for all $\ell = 0, \cdots, \theta$.*

Proof. Let $\boldsymbol{X}^\theta(0)$ be a random variable in $\mathbb{N}^{\theta+1}$ following the stationary distribution under the open-loop pricing policy $\hat{\boldsymbol{\varphi}}^\theta$. Let $\boldsymbol{X}^\theta(t)$ denote the state at time t under the open-loop pricing policy $\hat{\boldsymbol{\varphi}}^\theta$. Let $\boldsymbol{A}^\theta(t) \geq \mathbf{0}$ and $\boldsymbol{M}^\theta(t) \geq \mathbf{0}$ denote the corresponding load arrivals and bookings at time t respectively. We can express $X^\theta_\ell(t)$ for $t \geq \theta$ as

$$
\begin{aligned}
X^\theta_\ell(t) &= A^\theta_\ell(t) + X^\theta_{\ell+1}(t-1) - M^\theta_{\ell+1}(t-1) \\
&= \cdots \;=\; A^\theta_\ell(t) + \sum_{k=1}^{\theta-\ell} \left(A^\theta_{\ell+k}(t-k) - M^\theta_{\ell+k}(t-k)\right) \;\leq\; \sum_{k=0}^{\theta-\ell} A^\theta_{\ell+k}(t-k).
\end{aligned}
$$

Note that $A^\theta_{\ell+k}(t-k)$ is equal to 1 with probability $\lambda\psi^\theta_{\ell+k}$ and 0 otherwise. Additionally, the arrivals in different time periods are independent, so $\sum_{k=0}^{\theta-\ell} A^\theta_{\ell+k}(t-k)$ is the sum of independent binary variables (i.e., Poisson binomial distributed). Then, for any $q \geq 1$,

$$
\mathsf{E}\left[\left(X^\theta_\ell(t)\right)^q\right] \;\leq\; \mathsf{E}\left[\left(\sum_{k=0}^{\theta-\ell} A^\theta_{\ell+k}(t-k)\right)^q\right].
$$

Let $Z^\theta_\ell := \sum_{k=0}^{\theta-\ell} A^\theta_{\ell+k}(t-k)$. Using the result for Poisson binomial distribution (see, e.g. [105]), we have

$$
\begin{aligned}
\mathsf{E}\left[\sum_{k=0}^{\theta-\ell} A^\theta_{\ell+k}(t-k)\right] &= \sum_{k=\ell}^{\theta} \lambda\psi^\theta_k \;=\; \lambda\left(\sum_{k=\ell}^{\theta}\psi^\theta_k\right) \;\leq\; 1 \\
\mathsf{Var}\left(\left(\sum_{k=0}^{\theta-\ell} A^\theta_{\ell+k}(t-k)\right)^2\right) &= \sum_{k=\ell}^{\theta}\left(1-\lambda\psi^\theta_k\right)\lambda\psi^\theta_k \;\leq\; \sum_{k=\ell}^{\theta}\lambda\psi^\theta_k \;\leq\; 1.
\end{aligned}
$$

Then, we have

$$\mathsf{E}\left[\left(\sum_{k=0}^{\theta-\ell} A^{\theta}_{\ell+k}(t-k)\right)^2\right] = \mathsf{Var}\left(\left(\sum_{k=0}^{\theta-\ell} A^{\theta}_{\ell+k}(t-k)\right)^2\right) + \mathsf{E}\left[\sum_{k=0}^{\theta-\ell} A^{\theta}_{\ell+k}(t-k)\right]^2$$
$$\leq 1 + 1^2 = 2.$$

This implies that $\bar{x}^{\theta}_{\ell} = \mathsf{E}\left[X^{\theta}_{\ell}\right] \leq 1$ and $\mathsf{Var}(X^{\theta}_{\ell}) \leq \mathsf{E}\left[\left(X^{\theta}_{\ell}\right)^2\right] \leq 2$ for $\ell = 0, \ldots, \theta$. $\square$

B.4.2 Proof of Lemma 3.9

Lemma 3.9. *Suppose $\mu K_v < 1/4$ by Assumption 3.7. Let $\boldsymbol{X}^{\theta}$ denote the state vector under the stationary distribution of the open-loop pricing policy $\hat{\boldsymbol{\varphi}}^{\theta}$. For any $k = 1, \ldots, \theta$ and any sequence of constants $(a_{\ell}:\ 0 \leq a_{\ell} \leq 1,\ \ell = 1, \ldots, \theta)$, we have*

$$\mathsf{Var}\left(\sum_{\ell=k}^{\theta} a_{\ell} X^{\theta}_{\ell}\right) \ \leq \ K_c(\theta - k + 1), \tag{B.7}$$

where $K_c := 2/(1 - 4\mu K_v)$.

Proof. We prove the result by induction on the time index. Consider any initial state of the system $\boldsymbol{X}^{\theta}(0)$ that satisfies the condition (B.7) (for example, we can set $\boldsymbol{X}^{\theta}(0)$ to be any fixed state, which has zero variance). Note that the expectation used in the proof below is with respect to this initial distribution. We will show that for any time index t, if condition (B.7) holds for $\boldsymbol{X}^{\theta}(t)$, then it also holds for the state in the next period $\boldsymbol{X}^{\theta}(t+1)$. If this induction step is proved, the lemma immediately follows, because the state $\boldsymbol{X}^{\theta}(t)$ converges to the stationary distribution of the system (which is irreducible, aperiodic, and positive recurrent), namely $\boldsymbol{X}^{\theta}(t) \xrightarrow{\mathrm{d}} \boldsymbol{X}^{\theta}$, so $\boldsymbol{X}^{\theta}$ also satisfies the variance bound in (B.7).

Let $\boldsymbol{M}^{\theta}(t) = (M^{\theta}_{\ell}(t))$ be the random vector of bookings after drivers observing $\boldsymbol{X}^{\theta}(t)$ and the prices posted. Let $\boldsymbol{A}^{\theta}(t) = (A^{\theta}_{\ell}(t))$ be the random vector of newly arriving loads. The sequence $\left(\boldsymbol{A}^{\theta}(t), t \geq 1\right)$ is independent of $\left(\boldsymbol{X}^{\theta}(t), t \geq 1\right)$ and $\left(\boldsymbol{M}^{\theta}(t), t \geq 1\right)$. Given a current state $\boldsymbol{X}^\theta(t)$, the next system state $\boldsymbol{X}^\theta(t+1)$ is given by the dynamics:

$$
\begin{aligned}
X^\theta_{\ell-1}(t+1) \ &= \ X^\theta_\ell(t) - M^\theta_\ell(t) + A^\theta_{\ell-1}(t+1) \quad \text{for } \ell = 1, \ldots, \theta, \\
X^\theta_\theta(t+1) \ &= \ A^\theta_\theta(t+1).
\end{aligned}
$$

Note that

$$
\mathsf{E}\left[M^\theta_\ell(t) \mid \boldsymbol{X}^\theta(t)\right] \ = \ \mu \frac{\hat{v}^\theta_\ell X^\theta_\ell(t)}{\sum_{i=1}^\theta \hat{v}^\theta_i X^\theta_i(t) \ + \ \theta} \quad \text{for } \ell = 1, \ldots, \theta.
$$

Also, by the law of total covariance, note that

$$
\begin{aligned}
&\mathsf{Cov}\left(X^\theta_j(t),\ M^\theta_k(t)\right) \\
&= \ \mathsf{E}\left[\mathsf{Cov}\left(X^\theta_j(t),\ M^\theta_k(t) \mid \boldsymbol{X}^\theta(t)\right)\right] \ + \ \mathsf{Cov}\left(\mathsf{E}\left[X^\theta_j(t) \mid \boldsymbol{X}^\theta(t)\right],\ \mathsf{E}\left[M^\theta_k(t) \mid \boldsymbol{X}^\theta(t)\right]\right) \\
&= 0 \ + \ \mathsf{Cov}\left(X^\theta_j(t),\ \mathsf{E}\left[M^\theta_k(t) \mid \boldsymbol{X}^\theta(t)\right]\right) \\
&= \ \mathsf{Cov}\left(X^\theta_j(t),\ \mu \frac{\hat{v}^\theta_k X^\theta_k(t)}{\sum_{i=1}^\theta \hat{v}^\theta_i X^\theta_i(t) \ + \ \theta}\right).
\end{aligned}
\tag{B.8}
$$

Now, we prove the induction step. We assume the following induction hypothesis that at time t, for any sequence $(a_\ell :\ 0 \le a_\ell \le 1,\ \ell = 1, \ldots, \theta)$ and any $k = 1, \ldots, \theta$,

$$
\mathsf{Var}\left(\sum_{\ell=k}^{\theta} a_\ell X^\theta_\ell(t)\right) \ \le \ K_c\left(\theta - k + 1\right).
$$

We aim to show that at time $t+1$, for any sequence $(\tilde{a}_\ell :\ 0 \le \tilde{a}_\ell \le 1,\ \ell = 1, \ldots, \theta)$ and any $k = 1, \ldots, \theta$,

$$
\mathsf{Var}\left(\sum_{\ell=k}^{\theta} \tilde{a}_\ell X^\theta_\ell(t+1)\right) \ \le \ K_c\left(\theta - k + 1\right).
$$

Let $a_\ell = \tilde{a}_{\ell-1}$ for $\ell = 2, \ldots, \theta$. Then, for any $1 \le k \le \theta$,

$$\begin{aligned}\sum_{\ell=k}^{\theta} \tilde{a}_\ell X_\ell^\theta(t+1) &= \sum_{\ell=k}^{\theta-1} \tilde{a}_\ell \left(X_{\ell+1}^\theta(t) - M_{\ell+1}^\theta(t)\right) + \sum_{\ell=k}^{\theta} \tilde{a}_\ell A_\ell^\theta(t+1) \\ &= \sum_{\ell=k+1}^{\theta} a_\ell \left(X_\ell^\theta(t) - M_\ell^\theta(t)\right) + \sum_{\ell=k}^{\theta} \tilde{a}_\ell A_\ell^\theta(t+1).\end{aligned}$$

So,

$$\begin{aligned}\mathsf{Var}\left(\sum_{\ell=k}^{\theta} \tilde{a}_\ell X_\ell^\theta(t+1)\right) &\overset{(a)}{=} \mathsf{Var}\left(\sum_{\ell=k+1}^{\theta} a_\ell \left(X_\ell^\theta(t) - M_\ell^\theta(t)\right)\right) + \mathsf{Var}\left(\sum_{\ell=k}^{\theta} \tilde{a}_\ell A_\ell^\theta(t+1)\right) \\ &= \mathsf{Var}\left(\sum_{\ell=k+1}^{\theta} a_\ell X_\ell^\theta(t)\right) + \mathsf{Var}\left(\sum_{\ell=k+1}^{\theta} a_\ell M_\ell^\theta(t)\right) \\ &\quad + \mathsf{Var}\left(\sum_{\ell=k}^{\theta} \tilde{a}_\ell A_\ell^\theta(t+1)\right) \\ &\quad - 2\,\mathsf{Cov}\left(\sum_{\ell=k+1}^{\theta} a_\ell X_\ell^\theta(t),\ \sum_{\ell=k+1}^{\theta} a_\ell M_\ell^\theta(t)\right),\end{aligned}$$

where equality (a) holds due to the independence of $\boldsymbol{A}^\theta(t+1)$ from $\boldsymbol{X}^\theta(t)$ and $\boldsymbol{M}^\theta(t)$.

We examine the last three terms in the above equation. Note that $\sum_{\ell=k+1}^{\theta} M_\ell^\theta(t)$ represents the random booking of a load with lead time between $k+1$ and θ, which is Bernoulli distributed. So,

$$\mathsf{Var}\left(\sum_{\ell=k+1}^{\theta} a_\ell M_\ell^\theta(t)\right) \le \mathsf{E}\left[\left(\sum_{\ell=k+1}^{\theta} a_\ell M_\ell^\theta(t)\right)^2\right] \le \mathsf{E}\left[\left(\sum_{\ell=k+1}^{\theta} M_\ell^\theta(t)\right)^2\right] \le 1.$$

Similarly, note that $\sum_{\ell=k}^{\theta} A_\ell^\theta(t)$ represents the random arrival of a load with lead time between k and θ, which is Bernoulli distributed. So,

$$\mathsf{Var}\left(\sum_{\ell=k}^{\theta} \tilde{a}_\ell A_\ell^\theta(t)\right) \le \mathsf{E}\left[\left(\sum_{\ell=k}^{\theta} \tilde{a}_\ell A_\ell^\theta(t)\right)^2\right] \le \mathsf{E}\left[\left(\sum_{\ell=k}^{\theta} A_\ell^\theta(t)\right)^2\right] \le 1.$$

Lastly,

$$
\begin{aligned}
&\left|\mathsf{Cov}\left(\sum_{\ell=k+1}^{\theta} a_\ell X_\ell^\theta(t),\ \sum_{\ell=k+1}^{\theta} a_\ell M_\ell^\theta(t)\right)\right| \\
&\overset{(b)}{=} \left|\mathsf{Cov}\left(\sum_{\ell=k+1}^{\theta} a_\ell X_\ell^\theta(t),\ \mu\frac{\sum_{\ell=k+1}^{\theta} a_\ell \hat{v}_\ell^\theta X_\ell^\theta(t)}{\sum_{i=1}^{\theta} \hat{v}_i^\theta X_i^\theta(t)\ +\ \theta}\right)\right| \\
&\overset{(c)}{\leq} \mu\sqrt{\mathsf{Var}\left(\sum_{\ell=k+1}^{\theta} a_\ell X_\ell^\theta(t)\right)\cdot\mathsf{Var}\left(\frac{\sum_{\ell=k+1}^{\theta} a_\ell \hat{v}_\ell^\theta X_\ell^\theta(t)}{\sum_{i=1}^{\theta} \hat{v}_i^\theta X_i^\theta(t)\ +\ \theta}\right)}
\end{aligned}
$$

where equality (b) follows from (B.8), and inequality (c) applies the covariance inequality.

Next, we show that

$$
\mathsf{Var}\left(\frac{\sum_{\ell=k+1}^{\theta} a_\ell \hat{v}_\ell^\theta X_\ell^\theta(t)}{\sum_{i=1}^{\theta} \hat{v}_i^\theta X_i^\theta(t)\ +\ \theta}\right)\ \leq\ \frac{4K_cK_v^2}{\theta}.
$$

Let $Z_1^\theta := \sum_{\ell=k+1}^{\theta} a_\ell \hat{v}_\ell^\theta X_\ell^\theta(t)$ and $Z_2^\theta := \sum_{i=1}^{\theta} \hat{v}_i^\theta X_i^\theta(t) + \theta$. Then, $0 \leq Z_1^\theta/Z_2^\theta \leq 1$ w.p.1. Let $\mu_1^\theta := \mathsf{E}\left[Z_1^\theta\right]$ and $\mu_2^\theta := \mathsf{E}\left[Z_2^\theta\right] \geq \theta$. So,

$$
\begin{aligned}
\mathsf{Var}\left(\frac{Z_1^\theta}{Z_2^\theta}\right) &= \mathsf{E}\left[\left(\frac{Z_1^\theta}{Z_2^\theta} - \mathsf{E}\left[\frac{Z_1^\theta}{Z_2^\theta}\right]\right)^2\right] \ \overset{(d)}{\leq}\ \mathsf{E}\left[\left(\frac{Z_1^\theta}{Z_2^\theta} - \frac{\mu_1^\theta}{\mu_2^\theta}\right)^2\right] \\
&= \mathsf{E}\left[\left(\frac{Z_1^\theta}{Z_2^\theta} - \frac{Z_1^\theta}{\mu_2^\theta} + \frac{Z_1^\theta}{\mu_2^\theta} - \frac{\mu_1^\theta}{\mu_2^\theta}\right)^2\right] \\
&\leq 2\,\mathsf{E}\left[\left(\frac{Z_1^\theta}{Z_2^\theta} - \frac{Z_1^\theta}{\mu_2^\theta}\right)^2\right]\ +\ 2\,\mathsf{E}\left[\left(\frac{Z_1^\theta}{\mu_2^\theta} - \frac{\mu_1^\theta}{\mu_2^\theta}\right)^2\right] \\
&= 2\,\mathsf{E}\left[\left(\frac{Z_1^\theta\left(\mu_2^\theta - Z_2^\theta\right)}{Z_2^\theta\mu_2^\theta}\right)^2\right]\ +\ 2\,\frac{\mathsf{Var}\left(Z_1^\theta\right)}{\left(\mu_2^\theta\right)^2} \\
&\overset{(e)}{\leq} \frac{2\,\mathsf{Var}\left(Z_2^\theta\right)}{\left(\mu_2^\theta\right)^2}\ +\ \frac{2\,\mathsf{Var}\left(Z_1^\theta\right)}{\left(\mu_2^\theta\right)^2} \\
&\overset{(f)}{\leq} \frac{2K_cK_v^2\theta}{\theta^2}\ +\ \frac{2K_cK_v^2(\theta-k+1)}{\theta^2}\ \leq\ \frac{4K_cK_v^2}{\theta}.
\end{aligned}
$$

Inequality (d) uses the fact that for any constant a, $\mathsf{E}[(X-a)^2] = \mathsf{E}[(X-\mathsf{E}\,X)^2]+\mathsf{E}[(\mathsf{E}\,X-a)^2] \geq \mathsf{E}[(X-\mathsf{E}\,X)^2]$. Inequality (e) follows from $0 \leq Z_1^\theta/Z_2^\theta \leq 1$. Inequality (f) follows from the induction hypothesis. Therefore, our claim above holds.

Now, putting things together,

$$\begin{aligned}
\mathsf{Var}\left(\sum_{\ell=k}^{\theta} \tilde{a}_\ell X_\ell^\theta(t+1)\right) \;&\leq\; \mathsf{Var}\left(\sum_{\ell=k+1}^{\theta} a_\ell X_\ell^\theta(t)\right) \;+\; \mathsf{Var}\left(\sum_{\ell=k+1}^{\theta} a_\ell M_\ell^\theta(t)\right) \\
&\quad+\; \mathsf{Var}\left(\sum_{\ell=k}^{\theta} \tilde{a}_\ell A_\ell^\theta(t+1)\right) \\
&\quad+\; 2\left|\mathsf{Cov}\left(\sum_{\ell=k+1}^{\theta} a_\ell X_\ell^\theta(t),\; \sum_{\ell=k+1}^{\theta} a_\ell M_\ell^\theta(t)\right)\right| \\
&\leq K_c(\theta-k) \;+\; 2 \;+\; 2\mu\sqrt{K_c(\theta-k)\cdot\frac{4K_cK_v^2}{\theta}} \\
&\leq K_c(\theta-k) + 4\mu K_v K_c + 2.
\end{aligned}$$

If $4\mu K_v < 1$, we can set $K_c := \frac{2}{1-4\mu K_v}$. Then $4\mu K_v K_c + 2 = K_c$, so we have

$$\mathsf{Var}\left(\sum_{\ell=k}^{\theta} \tilde{a}_\ell X_\ell^\theta(t+1)\right) \;\leq\; K_c(\theta-k+1),$$

which is the desired conclusion.

As $t \to \infty$, we have $\boldsymbol{X}^\theta(t) \xrightarrow{\mathrm{d}} \boldsymbol{X}^\theta$, where $\boldsymbol{X}^\theta$ follows the stationary distribution of the Markov chain. Since the state space of this Markov chain is finite (since $X_\ell^\theta(t) \leq \theta$ for all $\ell = 1, \cdots, \theta$), it implies $\lim_{t\to\infty} \mathsf{Var}\left(\sum_{\ell=k}^{\theta} a_\ell X_\ell^\theta(t)\right) = \mathsf{Var}\left(\sum_{\ell=k}^{\theta} a_\ell X_\ell^\theta\right)$. Therefore, we have

$$\mathsf{Var}\left(\sum_{\ell=k}^{\theta} a_\ell X_\ell^\theta\right) \;\leq\; K_c\left(\theta-k+1\right),$$

for any $k = 1, \ldots, \theta$ and $\theta \geq 1$. $\square$

B.4.3 Auxiliary Lemma B.4

We prove a useful result to bound the expectation of the ratio of two random variables. The proof can be found in [72]. For completeness, we state it below.

Lemma B.4 (Expectation of the Ratio of Two Random Variables)**.** *Let Z_1 and Z_2 be two random variables with finite mean and variance. Let $\mu_1 = \mathsf{E}[Z_1], \mu_2 = \mathsf{E}[Z_2]$. Suppose there exists a constant $B > 0$ such that $0 \le \frac{Z_1}{Z_2} \le B$ a.s., then*

$$-\frac{\mathsf{Cov}(Z_1, Z_2)}{\mu_2^2} \;\le\; \mathsf{E}\left[\frac{Z_1}{Z_2}\right] \;-\; \frac{\mu_1}{\mu_2} \;\le\; -\;\frac{\mathsf{Cov}(Z_1, Z_2)}{\mu_2^2} \;+\; \frac{B\;\mathsf{Var}(Z_2)}{\mu_2^2}.$$

Proof. Noting the identity

$$\frac{1}{Z_2} \;=\; \frac{1}{\mu_2} \;-\; \frac{Z_2-\mu_2}{\mu_2\; Z_2},$$

we have

$$\begin{aligned}\mathsf{E}\left[\frac{Z_1}{Z_2}\right] &= \mathsf{E}\left[\frac{Z_1}{\mu_2} \;-\; \frac{Z_1\;(Z_2-\mu_2)}{\mu_2\; Z_2}\right] \;=\; \frac{\mu_1}{\mu_2} \;-\; \frac{1}{\mu_2}\mathsf{E}\left[\frac{Z_1\;(Z_2-\mu_2)}{Z_2}\right] \\ &\overset{(a)}{=} \frac{\mu_1}{\mu_2} \;-\; \frac{\mathsf{E}\left[Z_1\;(Z_2-\mu_2)\right]}{\mu_2^2} \;+\; \frac{1}{\mu_2^2}\mathsf{E}\left[\frac{Z_1\;(Z_2-\mu_2)^2}{Z_2}\right],\end{aligned}$$

where equality (a) recursively applies the identity.

Note that $\mathsf{E}\left[Z_1(Z_2-\mu_2)\right] = \mathsf{Cov}(Z_1, Z_2)$ and $\mathsf{E}\left[(Z_2-\mu_2)^2\right] = \mathsf{Var}(Z_2) > 0$. If $0 \le \frac{Z_1}{Z_2} \le B$ a.s., the last term is bounded by $B\,\mathsf{Var}(Z_2)/\mu_2^2$, which completes the proof. ☐

B.4.4 Proof of Lemma 3.10

Recall the choice probabilities in the MNL model is given by

$$f_0^\theta(\boldsymbol{p}^\theta, \boldsymbol{x}^\theta) = \frac{1}{\sum_{k=1}^{\theta} x_k^\theta\; v_k^\theta(p_k^\theta)/\theta \;+\; 1}, \quad f_\ell^\theta(\boldsymbol{p}^\theta, \boldsymbol{x}^\theta) = \frac{x_\ell^\theta\; v_\ell^\theta(p_\ell^\theta)/\theta}{\sum_{k=1}^{\theta} x_k^\theta\; v_k^\theta(p_k^\theta)/\theta \;+\; 1}, \quad \forall \ell \ge 1,$$

and the one period cost function is

$$r^\theta(\boldsymbol{p}^\theta, \boldsymbol{x}^\theta) \;=\; \sum_{\ell=1}^{\theta} \mu \frac{p_\ell^\theta\, x_\ell^\theta\, v_\ell^\theta(p_\ell^\theta)/\theta}{\sum_{k=1}^{\theta} x_k^\theta\, v_k^\theta(p_k^\theta)/\theta \;+\; 1} \;+\; C\, x_0^\theta.$$

Lemma 3.10. *There is a constant $K_f > 0$, such that*

$$\left| \sum_{\ell=k}^{\theta} \mathsf{E}\left[f_\ell^\theta(\hat{\boldsymbol{p}}^\theta, \boldsymbol{X}^\theta) \right] - \sum_{\ell=k}^{\theta} f_\ell^\theta(\hat{\boldsymbol{p}}^\theta, \bar{\boldsymbol{x}}^\theta) \right| \;\le\; \frac{K_f}{\theta}, \quad \textit{for } 1 \le k \le \theta,$$

and

$$\left| \mathsf{E}\left[r^\theta(\hat{\boldsymbol{p}}^\theta, \boldsymbol{X}^\theta) \right] - r^\theta(\hat{\boldsymbol{p}}^\theta, \bar{\boldsymbol{x}}^\theta) \right| \;\le\; \frac{\mu C K_f}{\theta},$$

where $K_f := 2K_v^2 K_c$.

Proof. We define

$$Z_k^\theta \;:=\; \sum_{\ell=k}^{\theta} \hat{v}_\ell^\theta\, X_\ell^\theta/\theta \quad \text{for } 1 \le k \le \theta, \quad \text{and} \quad Z^\theta \;:=\; \sum_{\ell=1}^{\theta} \hat{v}_\ell^\theta\, X_\ell^\theta/\theta \;=\; Z_1^\theta.$$

Then, we have

$$\mathsf{E}\left[\frac{Z_k^\theta}{Z^\theta + 1} \right] \;=\; \sum_{\ell=k}^{\theta} \mathsf{E}\left[f_\ell^\theta(\hat{\boldsymbol{p}}^\theta, \boldsymbol{X}^\theta) \right] \quad \text{and} \quad \frac{\mathsf{E}\left[Z_k^\theta \right]}{\mathsf{E}[Z^\theta] + 1} \;=\; \sum_{\ell=k}^{\theta} f_\ell^\theta(\hat{\boldsymbol{p}}^\theta, \bar{x}^\theta).$$

Since $0 \le Z_k^\theta / \big(Z^\theta + 1\big) \le 1$ w.p.1., by Lemma B.4, we have

$$-\frac{\mathsf{Cov}\left(Z_k^\theta, Z^\theta\right)}{\left(\mathsf{E}[Z^\theta] + 1\right)^2} \;\le\; \mathsf{E}\left[\frac{Z_k^\theta}{Z^\theta + 1} \right] \;-\; \frac{\mathsf{E}\left[Z_k^\theta \right]}{\mathsf{E}[Z^\theta] + 1} \;\le\; -\frac{\mathsf{Cov}\left(Z_k^\theta, Z^\theta\right)}{\left(\mathsf{E}[Z^\theta] + 1\right)^2} \;+\; \frac{\mathsf{Var}(Z^\theta)}{\left(\mathsf{E}[Z^\theta] + 1\right)^2}.$$

Next, we bound the terms $\mathsf{Cov}\left(Z_k^\theta, Z^\theta\right)$ and $\mathsf{Var}(Z^\theta)$ using Lemma 3.9. To using this Lemma, we note that the coefficients in the summand is bounded by $0 \le \hat{v}_\ell^\theta/\theta \le K_v/\theta$ for all $\ell = 1, \cdots, \theta$ by Lemma 3.5 Therefore, we have

$$\begin{aligned}
\mathsf{Cov}\left(Z_k^\theta, Z^\theta\right) &\le \sqrt{\mathsf{Var}(Z_k^\theta)\,\mathsf{Var}\left(Z^\theta\right)} \\
&\le \sqrt{\frac{(\theta - k + 1)K_v^2 K_c}{\theta^2} \cdot \frac{(\theta - 1 + 1)K_v^2 K_c}{\theta^2}} \;\le\; \frac{K_v^2 K_c}{\theta}, \\
\mathsf{Var}(Z^\theta) &\le \frac{(\theta - 1 + 1)K_v^2 K_c}{\theta^2} \;=\; \frac{K_v^2 K_c}{\theta}.
\end{aligned}$$

Then, by Lemma B.4, we have

$$\begin{aligned}
\left|\sum_{\ell=k_1}^{k_2} \left(\mathsf{E}\left[f_\ell^\theta(\hat{\boldsymbol{p}}^\theta, \boldsymbol{X}^\theta)\right] - f_\ell^\theta(\hat{\boldsymbol{p}}^\theta, \bar{\boldsymbol{x}}^\theta)\right)\right| &= \left|\mathsf{E}\left[\frac{Z_k^\theta}{Z^\theta + 1}\right] \;-\; \frac{\mathsf{E}\left[Z_k^\theta\right]}{\mathsf{E}[Z^\theta] + 1}\right| \\
&\le \left|\mathsf{Cov}\left(Z_k^\theta, Z^\theta\right)\right| \;+\; \left|\mathsf{Var}\left(Z^\theta\right)\right| \\
&\le \frac{2K_v^2 K_c}{\theta} \quad \text{for any } 1 \le k_1 \le k_2 \le \theta.
\end{aligned}$$

Let $K_f := 2K_v^2 K_c$, then the first part of the lemma is proved.

Now we prove the second part of the lemma using the same approach. Let

$$W^\theta \;:=\; \sum_{\ell=1}^{\theta} \hat{p}_\ell^\theta \; \hat{v}_\ell^\theta \; X_\ell^\theta / \theta.$$

Again, it follows from Lemma B.4 that

$$\begin{aligned}
\left|\mathsf{E}\left[r^\theta(\hat{\boldsymbol{p}}^\theta, \boldsymbol{X}^\theta)\right] - r^\theta(\hat{\boldsymbol{p}}^\theta, \bar{\boldsymbol{x}}^\theta)\right| &= \mu \left|\mathsf{E}\left[\frac{W^\theta}{Z^\theta + 1}\right] \;-\; \frac{\mathsf{E}\left[W^\theta\right]}{\mathsf{E}[Z^\theta] + 1}\right| \\
&\le \mu \left|\mathsf{Cov}\left(W^\theta, Z^\theta\right)\right| \;+\; \mu C \;\mathsf{Var}(Z^\theta).
\end{aligned}$$

Note that $\mathsf{E}[X_0^\theta] = \bar{x}_0^\theta$, so the last term of $\mathsf{E}\left[r^\theta(\hat{\boldsymbol{p}}^\theta, \boldsymbol{X}^\theta)\right]$ and $r^\theta(\hat{\boldsymbol{p}}^\theta, \bar{\boldsymbol{x}}^\theta)$ cancels out.

Since $\hat{p}_\ell^\theta \le C$ for any $\ell = 1, \ldots, \theta$ by Lemma 3.5, we have $0 \le W^\theta/(Z^\theta + 1) \le$

$\max_i \left\{\hat{p}_i^\theta\right\} \leq C$. As before, we have

$$\left|\mathsf{Cov}\left(W^\theta, Z^\theta\right)\right| \;\leq \sqrt{\mathsf{Var}(W^\theta)\,\mathsf{Var}\left(Z^\theta\right)} \;\leq\; \frac{CK_v^2K_c}{\theta}.$$

Therefore,

$$\begin{aligned}\left|\mathsf{E}\left[r^\theta(\hat{\boldsymbol{p}}^\theta, \boldsymbol{X}^\theta)\right] - r^\theta(\hat{\boldsymbol{p}}^\theta, \bar{\boldsymbol{x}}^\theta)\right| \;&\leq \mu \left|\mathsf{Cov}\left(W^\theta, Z^\theta\right)\right| \;+\; \mu C\;\mathsf{Var}(Z^\theta) \\ &\leq \frac{\mu CK_v^2K_c}{\theta} \;+\; \frac{\mu CK_v^2K_c}{\theta} \;\leq\; \frac{\mu CK_f}{\theta}.\end{aligned}$$

This completes the proof. $\square$

B.4.5 Proof of Lemma 3.11

The lemma bounds the difference between the DTFM solution, $\bar{\boldsymbol{x}}^\theta$, and the expected system state under the open-loop pricing policy generated from the DTFM solution, $\hat{\boldsymbol{x}}^\theta$.

Lemma 3.11. *There is a constant $K_x > 0$ such that for all $\ell = 0, 1, \cdots, \theta$,*

$$|\bar{x}_\ell^\theta - \hat{x}_\ell^\theta| \;\leq\; \frac{K_x}{\theta},$$

where $K_x := \mu(1 + K_v)K_f$.

Proof. The proof comprises of showing the following steps: (1) $\hat{\boldsymbol{x}}^\theta$ satisfies a system of equations (i.e., the constraints of DTFM); (2) $\bar{\boldsymbol{x}}^\theta$ satisfies the same equations plus some bounded perturbation on the right hand side; and (3) given the bound on the perturbation, we bound the sup-norm $\|\bar{\boldsymbol{x}}^\theta - \hat{\boldsymbol{x}}^\theta\|_\infty \leq \frac{K_x}{\theta}$ for some $K_x > 0$.

(1) Equations for $\hat{\boldsymbol{x}}^\theta$.

Since $(\hat{\boldsymbol{x}}^\theta, \hat{\boldsymbol{p}}^\theta)$ is an optimal solution for the θ-scaled DTFM, it satisfies

$$\hat{x}_\theta^\theta \;=\; \lambda\psi_\theta^\theta$$

$$\hat{x}_\ell^\theta \;-\; \hat{x}_{\ell-1}^\theta \;=\; \mu\frac{\hat{x}_\ell^\theta \; \hat{v}_\ell^\theta/\theta}{\sum_{k=1}^\theta \hat{x}_k^\theta \; \hat{v}_k^\theta/\theta + 1} \;-\; \lambda\psi_{\ell-1}^\theta \qquad \text{for } \ell = 1,\ldots,\theta.$$

Let

$$\hat{u}_0^\theta \;:=\; \frac{1}{\sum_{k=1}^\theta \hat{x}_k^\theta \hat{v}_k^\theta/\theta \;+\; 1} \quad \text{and} \quad \hat{u}_\ell^\theta \;:=\; \hat{x}_\ell^\theta \; \hat{v}_\ell^\theta \; \hat{u}_0^\theta \quad \text{for } \ell = 1,\ldots,\theta,$$

and

$$\hat{\boldsymbol{u}}^\theta := (\hat{u}_\ell^\theta, \ell = 1,\ldots,\theta).$$

Then $(\hat{\boldsymbol{x}}^\theta, \hat{\boldsymbol{u}}^\theta, \hat{u}_0^\theta)$ satisfies the following system of equations:

$$\hat{x}_\theta^\theta \;=\; \lambda\psi_\theta^\theta \tag{B.9a}$$

$$\hat{x}_\ell^\theta \;-\; \hat{x}_{\ell-1}^\theta \;=\; \mu\frac{\hat{u}_\ell^\theta}{\theta} \;-\; \lambda\psi_{\ell-1}^\theta \quad \text{for } \ell = 1,\ldots,\theta \tag{B.9b}$$

$$\hat{u}_\ell^\theta \;=\; \hat{u}_0^\theta \; \hat{v}_\ell^\theta \; \hat{x}_\ell^\theta \quad \text{for } \ell = 1,\ldots,\theta \tag{B.9c}$$

$$\frac{1}{\theta}\sum_{\ell=1}^\theta \hat{u}_\ell^\theta \;+\; \hat{u}_0^\theta \;=\; 1. \tag{B.9d}$$

(2) Equations for $\bar{\boldsymbol{x}}^\theta$.

Let

$$\epsilon_\ell^\theta \;:=\; \mu \; \frac{\bar{x}_\ell^\theta \; \hat{v}_\ell^\theta/\theta}{\sum_{k=1}^\theta \bar{x}_k^\theta \; \hat{v}_k^\theta/\theta + 1} \;-\; \mu\,\mathsf{E}\left[\frac{X_\ell^\theta \; \hat{v}_\ell^\theta/\theta}{\sum_{k=1}^\theta X_k^\theta \; \hat{v}_k^\theta/\theta + 1}\right].$$

Note that, by Lemma 3.10,

$$\left|\sum_{\ell=k}^\theta \epsilon_\ell^\theta\right| \leq \frac{\mu K_f}{\theta} \quad \text{for any } 1 \leq k \leq \theta.$$

Using a similar argument as in the proof of Theorem 3.4, we have

$$\begin{aligned}
\mathsf{E}[X^\theta_\theta] &= \lambda\psi^\theta_\theta, \\
\mathsf{E}[X^\theta_\ell] - \mathsf{E}[X^\theta_{\ell-1}] &= \mu\,\mathsf{E}\left[\frac{X^\theta_\ell\ \hat{v}^\theta_\ell/\theta}{\sum_{k=1}^\theta X^\theta_k\ \hat{v}^\theta_k/\theta+1}\right] - \lambda\psi^\theta_{\ell-1}, \qquad \text{for } \ell = 1,\dots,\theta.
\end{aligned}$$

Then, $\bar{\boldsymbol{x}}^\theta = \mathsf{E}[\boldsymbol{X}^\theta]$ satisfies

$$\begin{aligned}
\bar{x}^\theta_\theta &= \lambda\psi^\theta_\theta \\
\bar{x}^\theta_\ell - \bar{x}^\theta_{\ell-1} &= \mu\frac{\bar{x}^\theta_\ell\ \hat{v}^\theta_\ell/\theta}{\sum_{k=1}^\theta \bar{x}^\theta_k\ \hat{v}^\theta_k/\theta+1} - \lambda\psi^\theta_{\ell-1} - \epsilon^\theta_\ell, \qquad \text{for } \ell = 1,\dots,\theta.
\end{aligned}$$

Similarly, let

$$\bar{u}^\theta_0 := \frac{1}{\sum_{k=1}^\theta \bar{x}^\theta_k\hat{v}^\theta_k/\theta + 1} \quad \text{and} \quad \bar{u}^\theta_\ell := \bar{x}^\theta_\ell\ \hat{v}^\theta_\ell\ \bar{u}^\theta_0 \quad \text{for } \ell = 1,\dots,\theta,$$

and

$$\bar{\boldsymbol{u}}^\theta := (\bar{u}^\theta_\ell, \ell = 1,\dots,\theta).$$

Then $(\bar{\boldsymbol{x}}^\theta, \bar{\boldsymbol{u}}^\theta, \bar{u}^\theta_0)$ satisfies the system of equations:

$$\bar{x}^\theta_\theta = \lambda\psi^\theta_\theta \tag{B.10a}$$

$$\bar{x}^\theta_\ell - \bar{x}^\theta_{\ell-1} = \mu\frac{\bar{u}^\theta_\ell}{\theta} - \lambda\psi^\theta_{\ell-1} - \epsilon^\theta_\ell \quad \text{for } \ell = 1,\dots,\theta \tag{B.10b}$$

$$\bar{u}^\theta_\ell = \bar{u}^\theta_0\ \hat{v}^\theta_\ell\ \bar{x}^\theta_\ell \quad \text{for } \ell = 1,\dots,\theta \tag{B.10c}$$

$$\frac{1}{\theta}\sum_{\ell=1}^\theta \bar{u}^\theta_\ell + \bar{u}^\theta_0 = 1. \tag{B.10d}$$

(3) Bounding $\|\bar{\boldsymbol{x}}^\theta - \hat{\boldsymbol{x}}^\theta\|_\infty$

Let

$$\Delta_\ell^\theta \;:=\; \bar{x}_\ell^\theta - \hat{x}_\ell^\theta \quad \text{for } \ell = 0, 1, \ldots, \theta, \quad \text{and} \quad \delta^\theta \;:=\; \bar{u}_0^\theta - \hat{u}_0^\theta.$$

Then, trivially, $\Delta_\theta^\theta = \bar{x}_\theta^\theta - \hat{x}_\theta^\theta = 0$, because (B.9a) and (B.10a) are the same. We aim to bound δ^θ and $\left\{\Delta_\ell^\theta,\ \ell = 0, \ldots, \theta\right\}$.

Comparing terms in (B.9b) and (B.10b), we have

$$\begin{aligned}
\Delta_{\ell-1}^\theta &= \Delta_\ell^\theta \;-\; \frac{\mu}{\theta}\left(\bar{u}_\ell^\theta \;-\; \hat{u}_\ell^\theta\right) \;+\; \epsilon_\ell^\theta \\
&= \Delta_\ell^\theta \;-\; \frac{\mu}{\theta}\left(\bar{u}_0^\theta \hat{v}_\ell^\theta \bar{x}_\ell^\theta \;-\; \hat{u}_0^\theta\, \hat{v}_\ell^\theta\, \hat{x}_\ell^\theta\right) \;+\; \epsilon_\ell^\theta \\
&= \Delta_\ell^\theta \;-\; \frac{\mu}{\theta}\left(\bar{u}_0^\theta \hat{v}_\ell^\theta \left(\hat{x}_\ell^\theta \;+\; \Delta_\ell^\theta\right) \;-\; \left(\bar{u}_0^\theta \;-\; \delta^\theta\right)\, \hat{v}_\ell^\theta\, \hat{x}_\ell^\theta\right) \;+\; \epsilon_\ell^\theta \\
&= \Delta_\ell^\theta \;-\; \frac{\mu}{\theta}\left(\bar{u}_0^\theta \hat{v}_\ell^\theta \Delta_\ell^\theta \;+\; \delta^\theta\, \hat{v}_\ell^\theta\, \hat{x}_\ell^\theta\right) \;+\; \epsilon_\ell^\theta \\
&= \left(1 \;-\; \frac{\mu \hat{v}_\ell^\theta \bar{u}_0^\theta}{\theta}\right)\Delta_\ell^\theta \;-\; \frac{\mu \hat{v}_\ell^\theta \hat{x}_\ell^\theta}{\theta}\delta^\theta \;+\; \epsilon_\ell^\theta.
\end{aligned} \tag{B.11}$$

Similarly, comparing terms in (B.9d) and (B.10d), we have

$$\begin{aligned}
\delta^\theta &= \bar{u}_0^\theta \;-\; \hat{u}_0^\theta \;=\; -\sum_{\ell=1}^{\theta}\left(\frac{\bar{u}_\ell^\theta}{\theta} \;-\; \frac{\hat{u}_\ell^\theta}{\theta}\right) \\
&= -\frac{1}{\mu}\sum_{\ell=1}^{\theta}\left(\left(\bar{x}_\ell^\theta - \bar{x}_{\ell-1}^\theta + \epsilon_\ell^\theta\right) \;-\; \left(\hat{x}_\ell^\theta - \hat{x}_{\ell-1}^\theta\right)\right) \\
&= -\frac{1}{\mu}\sum_{\ell=1}^{\theta}\left(\Delta_\ell^\theta - \Delta_{\ell-1}^\theta + \epsilon_\ell^\theta\right) \\
&= \frac{1}{\mu}\left(\Delta_0^\theta - \sum_{\ell=1}^{\theta}\epsilon_\ell^\theta\right).
\end{aligned} \tag{B.12}$$

Assume θ is large enough such that $1 - \mu\hat{v}_\ell^\theta \bar{u}_0/\theta \geq 1 - \mu\hat{v}_\ell^\theta/\theta > 0$ for all $\ell = 1, \ldots, \theta$. (Note that $\hat{v}_\ell^\theta \leq K_v$ and $\mu \leq 1$, so it suffices to take $\theta \geq K_v$.) Now, we derive Δ_0^θ as a function of δ^θ from (B.11). We claim that

$$\Delta_\ell^\theta \;=\; \sum_{i=\ell+1}^{\theta} \left\{ \left[\prod_{j=\ell+1}^{i-1} \left(1 \;-\; \frac{\mu \hat{v}_j^\theta \bar{u}_0^\theta}{\theta} \right) \right] \; \left(\epsilon_i^\theta \;-\; \frac{\mu \hat{v}_i^\theta \hat{x}_i^\theta}{\theta} \delta^\theta \right) \right\} \quad \text{for } \ell = 0, \ldots, \theta. \quad \text{(B.13)}$$

This can be proved by induction on the index ℓ. For $\ell = \theta$, it can be verified that $\Delta_\theta^\theta = 0$ holds for the above equation. Then, assuming that the equation holds for ℓ, we prove for $\ell - 1$ that

$$\begin{aligned}
\Delta_{\ell-1}^\theta \;&=\; \left(1 - \frac{\mu \hat{v}_\ell^\theta \bar{u}_0^\theta}{\theta}\right) \Delta_\ell^\theta \;+\; \left(\epsilon_\ell^\theta - \frac{\mu \hat{v}_\ell^\theta \hat{x}_\ell^\theta}{\theta} \delta^\theta\right) \\
&= \left(1 - \frac{\mu \hat{v}_\ell^\theta \bar{u}_0^\theta}{\theta}\right) \; \sum_{i=\ell+1}^{\theta} \left\{ \left[\prod_{j=\ell+1}^{i-1} \left(1 - \frac{\mu \hat{v}_j^\theta \bar{u}_0^\theta}{\theta} \right) \right] \; \left(\epsilon_i^\theta - \frac{\mu \hat{v}_i^\theta \hat{x}_i^\theta}{\theta} \delta^\theta \right) \right\} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad + \; \left(\epsilon_\ell^\theta - \frac{\mu \hat{v}_\ell^\theta \hat{x}_\ell^\theta}{\theta} \delta^\theta\right) \\
&= \sum_{i=\ell+1}^{\theta} \left\{ \left[\prod_{j=\ell}^{i-1} \left(1 - \frac{\mu \hat{v}_j^\theta \bar{u}_0^\theta}{\theta} \right) \right] \; \left(\epsilon_i^\theta - \frac{\mu \hat{v}_i^\theta \hat{x}_i^\theta}{\theta} \delta^\theta \right) \right\} \;+\; \left(\epsilon_\ell^\theta - \frac{\mu \hat{v}_\ell^\theta \hat{x}_\ell^\theta}{\theta} \delta^\theta\right) \\
&= \sum_{i=\ell}^{\theta} \left\{ \left[\prod_{j=\ell}^{i-1} \left(1 - \frac{\mu \hat{v}_j^\theta \bar{u}_0^\theta}{\theta} \right) \right] \; \left(\epsilon_i^\theta - \frac{\mu \hat{v}_i^\theta \hat{x}_i^\theta}{\theta} \delta^\theta \right) \right\}.
\end{aligned}$$

Thus, by induction, equation (B.13) holds. Then,

$$\Delta_0^\theta \;=\; \sum_{i=1}^{\theta} \left\{ \left[\prod_{j=1}^{i-1} \left(1 \;-\; \frac{\mu \hat{v}_j^\theta \bar{u}_0^\theta}{\theta} \right) \right] \; \left(\epsilon_i^\theta \;-\; \frac{\mu \hat{v}_i^\theta \hat{x}_i^\theta}{\theta} \delta^\theta \right) \right\}.$$

Substituting the above expression of Δ_0^θ into (B.12) and arranging terms, we have

$$\begin{aligned}
\left|\delta^\theta\right| \;&=\; \left| \frac{\sum_{i=1}^{\theta} \epsilon_i^\theta \left[\prod_{j=1}^{i-1} \left(1 - \frac{\mu \hat{v}_j^\theta \bar{u}_0^\theta}{\theta} \right) - 1 \right]}{\mu \left\{ 1 \;+\; \sum_{i=1}^{\theta} \frac{\hat{v}_i^\theta \hat{x}_i^\theta}{\theta} \left[\prod_{j=1}^{i-1} \left(1 - \frac{\mu \hat{v}_j^\theta \bar{u}_0^\theta}{\theta} \right) \right] \right\}} \right| \\
&\leq \left| \frac{\sum_{i=1}^{\theta} \epsilon_i^\theta}{\mu} \right| \;\leq\; \frac{\mu K_f}{\theta} \cdot \frac{1}{\mu} = \frac{K_f}{\theta},
\end{aligned}$$

where the first inequality uses the fact that $\hat{x}_i^\theta \geq 0$ for any i, and the second inequality uses Lemma 3.10.

Then it follows from (B.13), and using Lemma 3.10 again, that

$$\left|\Delta_\ell^\theta\right| \;\leq\; \left|\sum_{i=\ell+1}^{\theta} \epsilon_i^\theta\right| + \left|\sum_{i=\ell+1}^{\theta} \frac{\mu \hat{v}_i^\theta \hat{x}_i^\theta}{\theta} \delta^\theta\right| \;\leq\; \frac{\mu(1+K_v)K_f}{\theta} \quad \text{for } \ell = 0, \ldots, \theta.$$

Letting $K_x := \mu(1+K_v)K_f$, we have the desired result that $|\bar{x}_\ell^\theta - \hat{x}_\ell^\theta| = |\Delta_\ell^\theta| \leq K_x/\theta$ for $\ell = 0, \ldots, \theta$ and for all $\theta \geq K_v$. □

B.5 Proof of Theorem 3.12 and Corollary 3.13

We define the following functions:

$$\begin{aligned}
\mathcal{F}(x(\tau),\ y(\tau),\ u(\tau),\ \tau) \;&=\; \frac{\mu}{\beta}\, u(\tau) \left(\ln\left(\frac{u(\tau)}{x(\tau)}\right) - b_0(\tau)\right) \\
\mathcal{G}_1(x(\tau),\ y(\tau),\ u(\tau),\ \tau) \;&=\; \mu\, u(\tau) - \lambda\, \phi(\tau) \\
\mathcal{G}_2(x(\tau),\ y(\tau),\ u(\tau),\ \tau) \;&=\; -\, u(\tau) \\
\mathcal{I}_1(x(\tau),\ \tau) \;&=\; C\, x(\tau) \\
\mathcal{I}_2(y(\tau),\ \tau) \;&=\; -\frac{\mu}{\beta}\, (1 - y(\tau)) \ln(y(\tau)).
\end{aligned}$$

The Hamiltonian function $\mathscr{H}$ is defined as

$$\begin{aligned}
\mathscr{H}(x(\tau),\ y(\tau),\ u(\tau),\ \pi_1(\tau),\ \pi_2(\tau),\ \tau) \;:=\; &\mathcal{F}(x(\tau),\ y(\tau),\ u(\tau),\ \tau) \\
&+ \pi_1(\tau)\, \mathcal{G}_1(x(\tau),\ y(\tau),\ u(\tau),\ \tau) + \pi_2(\tau)\, \mathcal{G}_2(x(\tau),\ y(\tau),\ u(\tau),\ \tau).
\end{aligned}$$

We can prove the optimality of a solution $(x^*,\ y^*,\ u^*)$ by showing the following (see, e.g., page 208, Theorem 2 in [75]):

A) The solution $(x^*(\tau),\ y^*(\tau),\ u^*(\tau))$ satisfies (3.16b)-(3.16e).

B) The Hamiltonian is differentiable and convex in $(x,\ y,\ u)$.

C) There exist differentiable costates π_1^* and π_1^* such that their dynamics are given by

$$\begin{aligned}\dot{\pi}_1^*(\tau) &= -\frac{\partial\, \mathscr{H}(x^*(\tau),\ y^*(\tau),\ u^*(\tau),\ \pi_1^*(\tau),\ \pi_2^*(\tau),\ \tau)}{\partial\, x}\\ \dot{\pi}_2^*(\tau) &= -\frac{\partial\, \mathscr{H}(x^*(\tau),\ y^*(\tau),\ u^*(\tau),\ \pi_1^*(\tau),\ \pi_2^*(\tau),\ \tau)}{\partial\, y}\end{aligned}$$

with the boundary conditions

$$\begin{aligned}\pi_1^*(0) &= -\frac{\partial\, \mathcal{I}_1(x^*(0),\ 0)}{\partial\, x} \quad \text{as } x(0) \text{ is free,}\\ \pi_2^*(1) &= \frac{\partial\, \mathcal{I}_2(y^*(1),\ 1)}{\partial\, y} \quad \text{as } y(1) \text{ is free.}\end{aligned}$$

And, we have

$$\frac{\partial\, \mathscr{H}(x^*(\tau),\ y^*(\tau),\ u^*(\tau),\ \pi_1^*(\tau),\ \pi_2^*(\tau),\ \tau)}{\partial\, u} = 0. \tag{B.14}$$

Theorem 3.12. *Suppose an optimal solution* $(x^*,\ y^*,\ u^*)$ *for problem* (3.16) *exists. Let* $(x^*,\ p^*)$ *be the corresponding optimal solution for the* CTFM*. We have the following statements:*

a) The solution $(x^*,\ y^*,\ u^*)$ *is a global optimal solution.*

b) There is a costate π_1^* *to the state equation* (3.16b) *such that* $\pi_1^*(\tau)$ *is increasing with* $\pi_1^*(0) = -C$.

c) There is a costate π_2^* *to the state equation* (3.16c) *such that* $\pi_2^* \equiv \frac{1-u_0^*}{u_0^*} - \ln(u_0^*)$ *for some* $u_0^* \in [0, 1]$.

d) The optimal price trajectory p^* *is decreasing.*

Proof of Theorem 3.12. Examining the following first- and second-order derivative of the Hamiltonian $\mathscr{H}$, we have

$$\begin{aligned}
\frac{\partial\,\mathscr{H}(x(\tau),\ y(\tau),\ u(\tau),\ \pi_1(\tau),\ \pi_2(\tau),\ \tau)}{\partial\,x} &= -\,\frac{\mu\,u(\tau)}{\beta\,x(\tau)}\\
\frac{\partial\,\mathscr{H}(x(\tau),\ y(\tau),\ u(\tau),\ \pi_1(\tau),\ \pi_2(\tau),\ \tau)}{\partial\,u} &= \frac{\mu}{\beta}\left(\ln\left(\frac{u(\tau)}{x(\tau)}\right) - b_0(\tau) + 1\right)\\
&\qquad + \mu\pi_1(\tau) - \pi_2(\tau)\\
\frac{\partial^2\,\mathscr{H}(x(\tau),\ y(\tau),\ u(\tau),\ \pi_1(\tau),\ \pi_2(\tau),\ \tau)}{\partial\,x^2} &= \frac{\mu\,u(\tau)}{\beta\,x^2(\tau)}\ >\ 0\\
\frac{\partial^2\,\mathscr{H}(x(\tau),\ y(\tau),\ u(\tau),\ \pi_1(\tau),\ \pi_2(\tau),\ \tau)}{\partial\,u^2} &= \frac{\mu}{\beta\,u(\tau)}\ >\ 0\\
\frac{\partial^2\,\mathscr{H}(x(\tau),\ y(\tau),\ u(\tau),\ \pi_1(\tau),\ \pi_2(\tau),\ \tau)}{\partial\,x\,\partial\,u} &= -\,\frac{\mu}{\beta\,x(\tau)}.
\end{aligned}$$

Noting that

$$\begin{bmatrix} \frac{\partial^2\,\mathscr{H}}{\partial\,x^2} & \frac{\partial^2\,\mathscr{H}}{\partial\,x\,\partial\,u} \\[2ex] \frac{\partial^2\,\mathscr{H}}{\partial\,x\,\partial\,u} & \frac{\partial^2\,\mathscr{H}}{\partial\,u^2} \end{bmatrix} = \begin{bmatrix} \frac{\mu\,u}{\beta\,x^2} & -\frac{\mu}{\beta\,x} \\[2ex] -\frac{\mu}{\beta\,x} & \frac{\mu}{\beta\,u} \end{bmatrix} \text{ is positive semidefinite,}$$

and that state y is not presented in the Hamiltonian $\mathscr{H}$, an optimal solution $(x^*,\ y^*,\ u^*)$ to the problem is guaranteed to be a global optimal solution. So, statement a) holds.

Noting that the dynamics of the costate π_1^* is given by

$$\dot{\pi}_1^*(\tau)\ =\ \frac{\mu\,u^*(\tau)}{\beta\,x^*(\tau)}\ \geq\ 0,$$

we conclude that the costate π_1^* is monotonically increasing. Also, we have

$$\pi_1^*(0)\ =\ -\,\frac{\partial\,\mathcal{I}_1(x^*(0),\ 0)}{\partial\,x}\ =\ -\,C.$$

So, statement b) holds.

Noting that state y is not presented in the Hamiltonian $\mathscr{H}$, we have $\dot{\pi}_2^*(\tau)\ \equiv\ 0$, which implies that π_2^* is a constant. Then, we have

$$\pi_2^* \equiv \pi_2^*(1) = \frac{\partial\, \mathcal{I}_1(u^*(1),\ 1)}{\partial\, x} = \frac{\mu}{\beta}\left[\ln\left(u^*(1)\right) - \frac{1-u^*(1)}{u^*(1)}\right] = \frac{\mu}{\beta}\left[\ln\left(u_0^*\right) - \frac{1-u_0^*}{u_0^*}\right],$$

letting $u^*(1) = u_0^*$. So, statement c) follows.

Replacing π_2^* with the above expression in (B.14), arranging terms and noting the connection (3.17), we have

$$p^*(\tau) = \frac{1}{\beta}\left(\ln\left(\frac{u^*(\tau)}{x^*(\tau)\ u_0^*}\right) - b_0(\tau)\right) = -\,\pi_1^*(\tau) - \frac{1}{\beta u_0^*}.$$

Then we have statement d) as $\pi_1^*(\tau)$ is increasing. □

Corollary 3.13. *In the Homogeneous Preference setting, i.e.,* $b_0(\tau) \equiv \beta_0$*, and with all new loads arriving with lead time equal to 1, i.e.,* $\Phi(1-) = 0$*, an optimal solution* $(x^*,\ y^*,\ u^*)$*, if existing, for problem* (3.16) *is given in the following form:*

$$x^*(\tau) = [\lambda - \mu(1-u_0^*)] + \mu(1-u_0^*)\tau,\ y^*(\tau) = 1-(1-u_0^*)\tau,\ \textit{and}\ u^*(\tau) = 1-u_0^*,$$

with the costate variables

$$\pi_1^*(\tau) = \frac{1}{\beta}\ln\left(\frac{[\lambda-\mu(1-u_0^*)]+\mu(1-u_0^*)\tau}{\lambda-\mu(1-u_0^*)}\right) - C\ \textit{and}\ \pi_2^* \equiv \frac{\mu}{\beta}\left[\ln\left(u_0^*\right) - \frac{1-u_0^*}{u_0^*}\right],$$

where $u_0^* \in (0,1)$ *and* $\lambda > \mu(1-u_0^*)$*. And,* u_0^* *is determined by solving a convex program.*

Proof of Corollary 3.13. The first part of the statement is easy to validate.

Noting that $b_0 \equiv \beta_0$, we write the objective of problem (3.16) as a function of u_0:

$$\begin{aligned} J(u_0) &= \frac{\mu}{\beta}\int_0^1 (1-u_0)\left(\ln\left(\frac{1-u_0}{[\lambda-\mu(1-u_0)]+\mu(1-u_0)\tau}\right) - \beta_0\right)\mathrm{d}\tau \\ &\qquad - \frac{\mu}{\beta}(1-u_0)\ln(u_0) + C(\lambda-\mu(1-u_0)) \end{aligned}$$

$$
\begin{aligned}
&= \frac{\mu}{\beta}(1-u_0)\left(\ln(1-u_0)-\beta_0\right) \;-\; \frac{\mu}{\beta}\int_0^1 (1-u_0)\ln\left([\lambda-\mu(1-u_0)]+\mu(1-u_0)\tau\right)\mathrm{d}\tau \\
&\qquad\qquad -\; \frac{\mu}{\beta}(1-u_0)\ln(u_0) \;+\; C(\lambda-\mu(1-u_0)) \\
&= C\,\lambda \;+\; \frac{\mu}{\beta}(1-u_0)\left(\ln\left(\frac{1-u_0}{u_0}\right)-\beta C-\beta_0\right) \\
&\qquad\qquad -\; \frac{\mu}{\beta}\,(1-u_0)\;\left[\frac{\lambda-\mu(1-u_0)(1-\tau)}{\mu(1-u_0)}\;\ln\left(\lambda-\mu(1-u_0)(1-\tau)\right)-\tau\right]_0^1 \\
&= C\,\lambda \;+\; \frac{\mu}{\beta}(1-u_0)\left(\ln\left(\frac{1-u_0}{u_0}\right)-\beta C-\beta_0+1\right) \\
&\qquad\qquad -\; \frac{1}{\beta}\left[\lambda\ln(\lambda)-(\lambda-\mu(1-u_0))\ln(\lambda-\mu(1-u_0))\right].
\end{aligned}
$$

Note that the domain of the function is $\{u_0 \;:\; 0<u_0<1,\; \lambda>\mu(1-u_0)\}$. Then

$$
\begin{aligned}
\frac{\mathrm{d}J(u_0)}{\mathrm{d}u_0} &= \frac{\mu}{\beta}\left[\beta C+\beta_0-1-\ln\left(\frac{1-u_0}{u_0}\right)-\frac{1}{u_0}+\ln(\lambda-\mu(1-u_0))+1\right] \\
&= \frac{\mu}{\beta}\left[\beta C+\beta_0-\ln\left(\frac{1-u_0}{u_0}\right)-\frac{1}{u_0}+\ln(\lambda-\mu(1-u_0))\right], \\
\frac{\mathrm{d}^2J(u_0)}{\mathrm{d}u_0^2} &= \frac{\mu}{\beta}\left[\frac{1}{u_0(1-u_0)}+\frac{1}{u_0^2}+\frac{\mu}{\lambda-\mu(1-u_0)}\right] \;>\; 0.
\end{aligned}
$$

So, the optimal u_0^* is determined by solving the convex program $\{\min_{u_0}\; J(u_0)\}$. □

B.6 Structure of the Optimal Price in Each State of the MDP

Recall the optimization problem in state $\boldsymbol{x}$, given the corresponding displacement cost $\boldsymbol{\Delta}$ and considering $\mathcal{P}=\mathbb{R}_+^L$:

$$
\min_{(\boldsymbol{u},u_0)\in\mathbb{R}_+^{L+1}} \quad \frac{\mu}{\beta}\sum_{\ell\in\text{supp}(\boldsymbol{x})} u_\ell\left(\ln\left(\frac{u_\ell}{x_\ell}\right)-\beta\Delta_\ell^\theta-\beta_\ell^0\right)-\frac{\mu}{\beta}(1-u_0)\ln(u_0) \tag{B.15a}
$$

$$
\text{s.t.} \qquad \sum_{\ell\in\text{supp}(\boldsymbol{x})} u_\ell+u_0 \;=\; 1 \tag{B.15b}
$$

$$
\mu u_\ell \;\geq\; \mu\exp(\beta_\ell^0)x_\ell u_0 \qquad \text{for } \ell\in\text{supp}(\boldsymbol{x}), \tag{B.15c}
$$

where we have a multiplier μ on both sides of constraint (B.15c) for convenience. Letting $(\boldsymbol{u}^*, u_0^*)$ be the optimal solution to the above problem, we can compute the corresponding optimal price $\boldsymbol{p}^*$ in state $\boldsymbol{x}$ by

$$p_\ell^* \;=\; \frac{1}{\beta}\left(\ln\left(\frac{u_\ell^*}{x_\ell u_0^*}\right) - \beta_\ell^0\right) \quad \text{for } \ell \in \operatorname{supp}(\boldsymbol{x}). \tag{B.16}$$

We let ζ be the Lagrangian multiplier associated with constraint (B.15b) and σ_ℓ be the Lagrangian multiplier associated with the nonnegativity constraint (B.15c) on price p_ℓ for $\ell \in \operatorname{supp}(\boldsymbol{x})$, and denote by $\boldsymbol{\sigma} = (\sigma_\ell)$. Then, by the KKT condition, the necessary and sufficient condition for $(\boldsymbol{u}^*, u_0^*)$ to be optimal to problem (3.5) is that there exists Lagrangian multipliers $(\zeta^*, \boldsymbol{\sigma}^*)$ such that

$$\frac{\mu}{\beta}\left(\ln\left(\frac{u_\ell^*}{x_\ell}\right) - \beta(\Delta_\ell + \sigma_\ell^*) - \beta_\ell^0 + 1\right) - \zeta^* \;=\; 0 \qquad \text{for } \ell \in \operatorname{supp}(\boldsymbol{x}), \tag{B.17a}$$

$$\frac{\mu}{\beta}\left(\ln(u_0^*) - \frac{1-u_0^*}{u_0^*}\right) - \zeta^* + \mu \sum_{\ell \in \operatorname{supp}(\boldsymbol{x})} \exp(\beta_\ell^0) x_\ell \sigma_\ell^* \;=\; 0 \tag{B.17b}$$

$$\sum_{\ell \in \operatorname{supp}(\boldsymbol{x})} u_\ell^* + u_0^* \;=\; 1 \tag{B.17c}$$

$$\mu u_\ell^* \;\geq\; \mu \exp(\beta_\ell^0) x_\ell^* u_0^* \qquad \text{for } \ell \in \operatorname{supp}(\boldsymbol{x}), \tag{B.17d}$$

$$\sigma_\ell^*(\mu u_\ell^* - \mu \exp(\beta_\ell^0) x_\ell^* u_0^*) \;=\; 0 \qquad \text{for } \ell \in \operatorname{supp}(\boldsymbol{x}), \tag{B.17e}$$

$$u_0^* \geq 0, \quad u_\ell^* \geq 0, \quad \sigma_\ell^* \geq 0 \qquad \text{for } \ell \in \operatorname{supp}(\boldsymbol{x}).$$

Combining (B.17a) and (B.17b) and eliminating ζ^*, we have

$$\frac{\mu}{\beta}\left(\ln\left(\frac{u_\ell^*}{x_\ell u_0^*}\right) - \beta(\Delta_\ell + \sigma_\ell^* + \varpi(\boldsymbol{\sigma}^*)) - \beta_\ell^0 + 1 + \frac{1-u_0^*}{u_0^*}\right) \;=\; 0 \quad \text{for } \ell \in \operatorname{supp}(\boldsymbol{x}), \tag{B.18}$$

where $\varpi(\boldsymbol{\sigma}) \;=\; \sum_{\ell \in \operatorname{supp}(\boldsymbol{x})} \exp(\beta_\ell^0) x_\ell \sigma_\ell^*$. Arranging terms in the above expression and taking exponentiation on both sides, we have

$$\frac{u_\ell^*}{u_0^*}\exp\left(\frac{1-u_0^*}{u_0^*}\right) \;=\; x_\ell \exp(\beta(\Delta_\ell^\theta + \sigma_\ell^* + \varpi(\boldsymbol{\sigma}^*)) + \beta_\ell^0 - 1) \quad \text{for } \ell \in \operatorname{supp}(\boldsymbol{x}).$$

Summing both sides over $\ell \in \operatorname{supp}(\boldsymbol{x})$ and noting (B.17c), we have

$$\begin{aligned} \frac{1-u_0^*}{u_0^*}\exp\left(\frac{1-u_0^*}{u_0^*}\right) \;&=\; \sum_{\ell\in\operatorname{supp}(\boldsymbol{x})} x_\ell \exp(\beta(\Delta_\ell^\theta + \sigma_\ell^* + \varpi(\boldsymbol{\sigma}^*)) + \beta_\ell^0 - 1) \\ \Rightarrow \quad \frac{1-u_0^*}{u_0^*} \;&=\; W\left(\sum_{\ell\in\operatorname{supp}(\boldsymbol{x})} x_\ell \exp(\beta(\Delta_\ell^\theta + \sigma_\ell^* + \varpi(\boldsymbol{\sigma}^*)) + \beta_\ell^0 - 1)\right), \end{aligned}$$

where $W(\cdot)$ is the Lambert W function. Leveraging the above result, we have the optimal price $\boldsymbol{p}^*$ to problem satisfy the following

$$\begin{aligned} p_\ell^* &\overset{(a)}{=} \frac{1}{\beta}\left(\ln\left(\frac{u_\ell^*}{x_\ell u_0^*} - \beta_\ell^0\right)\right) \\ &\overset{(b)}{=} (\Delta_\ell + \sigma_\ell^* + \varpi(\boldsymbol{\sigma}^*)) - \frac{1}{\beta}\left(W\left(\sum_{k\in\operatorname{supp}(\boldsymbol{x})} x_k \exp(\beta(\Delta_k + \sigma_k^* + \varpi(\boldsymbol{\sigma}^*)) + \beta_k^0 - 1)\right) + 1\right) \end{aligned}$$

for all $\ell \in \operatorname{supp}(\boldsymbol{x})$, where equality (a) follows from (B.16) and equality (b) follows by arranging terms in (B.18). Noting that complementary slackness (B.17e) can also be expressed as $p_\ell^*\sigma_\ell^* = 0$, we observe that the optimal price $\boldsymbol{p}^*$ follows the same order as the displacement cost $\boldsymbol{\Delta}$ in state $\boldsymbol{x}$.

When the optimal price $\boldsymbol{p}^*$ is strictly positive and there is at most one load of each lead time, i.e., $p_\ell^* > 0$ and $x_\ell \in \{0,1\}$ for all $\ell \in \operatorname{supp}(\boldsymbol{x})$, the expression of the optimal price is simplified to

$$p_\ell^* \;=\; \Delta_\ell - \frac{1}{\beta}\left(W\left(\sum_{k\in\operatorname{supp}(\boldsymbol{x})} \exp(\beta\Delta_k + \beta_\ell^0 - 1)\right) + 1\right) \qquad \text{for } \ell \in \operatorname{supp}(\boldsymbol{x}).$$

In [46], the authors considered a revenue maximization problem and derived a similar expression as the above one for the optimal price. Our approach of deduction also works for their setting and incorporates constraints on price.

B.7 Optimal Dynamic Pricing in the Heterogeneous Preference Setting

The example below shows that in the Heterogeneous Preference setting, it may not be optimal to post a higher price for loads with a shorter lead time.

Example B.5. *The number of lead times is $L = 3$. The probabilities of arrivals of new loads and drivers within each period are given by $\lambda = \mu = 0.2$. All new loads arrives with initial lead time L. In the multinomial logit model, the price sensitivity coefficient is $\beta = 1$ and the non-monetary utilities for loads of different lead times are given by $\boldsymbol{\beta}^0 = (-5, 1, -2)$. The penalty cost of each expiring load is $C = 10$. We let $\boldsymbol{p}^* : \{0,1\}^4 \mapsto \mathbb{R}_+^3$ denote the optimal pricing scheme for the MDP. Computing the optimial pricing scheme, we have $p_2^*(\boldsymbol{x}) = 1.0564 > p_1^*(\boldsymbol{x}) = 1.0552$ for state $\boldsymbol{x} = (0, 1, 1, 1)$.*

APPENDIX C

APPENDIX FOR CHAPTER 4

C.1 Comparison with the Markov Chain Choice Model

We give an example to show that our mixture of multinomial logit and independent demand models is not a special case of the Markov chain choice model. Under the Markov chain choice model, a customer arriving into the system is interested in purchasing product i with probability γ_i. If this product is available for purchase, then the customer purchases it. Otherwise, the customer transitions from product i to product j with probability ρ_{ij} and checks whether product j is available for purchase. With probability $1 - \sum_{j \in N} \rho_{ij}$, the customer transitions to the no-purchase option, in which case, she leaves without a purchase. In this way, the customer transitions among the products according to a Markov chain until she visits a product that is available for purchase or she visits the no-purchase option. The parameters of the Markov chain choice model are $\{\gamma_i : i \in N\}$ and $\{\rho_{ij} : i, j \in N\}$. Given that we offer the assortment $S \subseteq N$ of products, we let $P_i(S)$ be the expected number of times that a customer visits product i during the course of her choice process. If $i \in S$, then a customer purchases product i as soon as she visits this product, so for $i \in S$, $P_i(S)$ is the purchase probability of product i when we offer the assortment S. We can compute $\{P_i(S) : i \in N\}$ by solving the system of equations

$$P_i(S) = \gamma_i + \sum_{j \notin S} \rho_{ji} \, P_j(S) \qquad \forall \, i \in N. \tag{C.1}$$

We can intuitively justify (C.1) through a balance argument [30]. On the left side, $P_i(S)$ is the expected number of times that a customer visits product i during the course of her choice process. For a customer to visit product i, she may arrive into the system with an interest to purchase product i, which happens with probability γ_i. Alternatively, she may visit some product $j \notin S$ and the expected number of visits to this product is $P_j(S)$. In this case, if she transitions from product j to product i, then the customer ends up visiting product i. The probability of transitioning from product j to product i is ρ_{ji}. If $\sum_{j \in N} \rho_{ij} < 1$ for all $i \in N$, then there exists a solution to the system of equations above for any $S \subseteq N$.

We consider an instance of the mixture of multinomial logit and independent demand models with $N = \{1, 2, 3\}$, $(v_1, v_2, v_3) = (1, 1, 1)$, $(\theta_1, \theta_2, \theta_3) = (0, 0, 1)$ and $\beta = \frac{3}{4}$. Under this choice model, if we offer the assortment S, then a customer purchases product $i \in S$ with probability $\phi_i(S) = \beta \, \frac{v_i}{1+V(S)} + (1 - \beta)\, \theta_i$. Note that we did not normalize the size of the first customer segment to one. In Table C.1, we give the choice probabilities $\{\phi_i(S) : i \in S,\ S \subseteq N\}$ for this instance of the mixture of multinomial logit and independent demand models. We argue that there exists no Markov chain choice model such that the choice probabilities under the Markov chain choice model for all products and for all assortments match those under the mixture of multinomial logit and independent demand models. In other words, there exist no parameters $\{\gamma_i : i \in N\}$ and $\{\rho_{ij} : i, j \in N\}$ for the Markov chain choice model such that $P_i(S) = \phi_i(S)$ for all $S \subseteq N$, $i \in N$. To make this argument, by (C.1), note that $P_i(\{1, 2, 3\}) = \gamma_i$ for all $i \in N$. Thus, to ensure that $P_i(\{1, 2, 3\}) = \phi_i(\{1, 2, 3\})$ for all $i \in N$, we must choose $\{\gamma_i : i \in N\}$ so that $\gamma_1 = \phi_1(\{1, 2, 3\}) = \frac{3}{16}$, $\gamma_2 = \phi_2(\{1, 2, 3\}) = \frac{3}{16}$ and $\gamma_3 = \phi_3(\{1, 2, 3\}) = \frac{7}{16}$, fixing the values of the parameters $\{\gamma_i : i \in N\}$.

Consider an assortment of the form $N \setminus \{i\}$. Product i is the only one not in the assortment $N \setminus \{i\}$, so by (C.1), we get $P_k(N \setminus \{i\}) = \gamma_k + \rho_{ik}\, P_i(N \setminus \{i\})$ for all $k \in N$. Using the last equality with $k = i$, we get $(1 - \rho_{ii})\; P_i(N \setminus \{i\}) = \gamma_i$, so the equality $P_k(N \setminus \{i\}) = \gamma_k + \rho_{ik}\, P_i(N \setminus \{i\})$ is equivalent to $P_k(N \setminus \{i\}) = \gamma_k + \rho_{ik}\, \frac{\gamma_i}{1-\rho_{ii}}$, which, in turn, is equivalent to $\rho_{ik} = \frac{1-\rho_{ii}}{\gamma_i}\,(P_k(N \setminus \{i\}) - \gamma_k)$. Thus, to ensure that $P_k(N \setminus \{i\}) = \phi_k(N \setminus \{i\})$ for all $i, k \in N$, we must have

$$\rho_{ik} = \frac{1 - \rho_{ii}}{\gamma_i}\,(\phi_k(N \setminus \{i\}) - \gamma_k).$$

Table C.1: Expected revenue provided by all possible assortments

S	$\phi_1(S)$	$\phi_2(S)$	$\phi_3(S)$	S	$\phi_1(S)$	$\phi_2(S)$	$\phi_3(S)$
$\varnothing$	0	0	0	$\{1,2\}$	1/4	1/4	0
$\{1\}$	3/8	0	0	$\{1,3\}$	1/4	0	1/2
$\{2\}$	0	3/8	0	$\{2,3\}$	0	1/4	1/2
$\{3\}$	0	0	5/8	$\{1,2,3\}$	3/16	3/16	7/16

Using the values of $\phi_k(N \setminus \{i\})$ for $i, k \in N$ in Table C.1 and the fact that $\gamma_1 = \frac{3}{16}$, $\gamma_2 = \frac{3}{16}$ and $\gamma_3 = \frac{3}{16}$, the expression above yields $\rho_{21} = \frac{1}{3}(1 - \rho_{22})$, $\rho_{31} = \frac{1}{7}(1 - \rho_{33})$, $\rho_{23} = \frac{1}{3}(1 - \rho_{22})$ and $\rho_{32} = \frac{1}{7}(1 - \rho_{33})$. Lastly, consider the assortment $\{1\}$. By (C.1), we have $P_2(\{1\}) = \gamma_2 + \rho_{22}\, P_2(\{1\}) + \rho_{32}\, P_3(\{1\})$, which is equivalent to $(1 - \rho_{22})\, P_2(\{1\}) = \gamma_2 + \rho_{32}\, P_3(\{1\})$. Similarly, $(1 - \rho_{33})\, P_3(\{1\}) = \gamma_3 + \rho_{23}\, P_2(\{1\})$. Since $\rho_{23} = \frac{1}{3}(1 - \rho_{22})$ and $\rho_{32} = \frac{1}{7}(1 - \rho_{33})$, the last two equalities become

$$\begin{aligned} (1 - \rho_{22})\, P_2(\{1\}) &= \gamma_2 + \tfrac{1}{7}\,(1 - \rho_{33})\, P_3(\{1\}) \\ (1 - \rho_{33})\, P_3(\{1\}) &= \gamma_3 + \tfrac{1}{3}\,(1 - \rho_{22})\, P_2(\{1\}). \end{aligned}$$

Since $\gamma_2 = \frac{3}{16}$ and $\gamma_3 = \frac{7}{16}$, solving the equalities above, we get $(1 - \rho_{22})\, P_2(\{1\}) = \frac{21}{80}$ and $(1 - \rho_{33})\, P_3(\{1\}) = \frac{21}{40}$. Also, by (C.1), we have $P_1(\{1\}) = \gamma_1 + \rho_{21}\, P_2(\{1\}) + \rho_{31}\, P_3(\{1\})$. Noting that $\gamma_1 = \frac{3}{16}$, $\rho_{21} = \frac{1}{3}(1 - \rho_{22})$ and $\rho_{31} = \frac{1}{7}(1 - \rho_{33})$, we get $P_1(\{1\}) = \frac{3}{16} + \frac{1}{3}(1 - \rho_{22})\, P_2(\{1\}) + \frac{1}{7}(1 - \rho_{33})\, P_3(\{1\})$, but since $(1 - \rho_{22})\, P_2(\{1\}) = \frac{21}{80}$ and $(1 - \rho_{33})\, P_3(\{1\}) = \frac{21}{40}$, plugging them in the last equality, we must have have $P_1(\{1\}) = \frac{7}{20}$, which is different from $\phi_1(\{1\}) = \frac{3}{8}$.

Thus, we cannot choose the parameters of the Markov chain choice model to make sure that its choice probabilities match those in Table C.1. The example that we give in this section is not hard to find. Virtually for all randomly generated instances of our choice model, we cannot calibrate a Markov chain choice model to match the choice probabilities of our choice model.

C.2 Proof of Lemma 4.1

Let $H = \{i \in N : \hat{x}_i = \hat{x}_0\}$, $M = \{i \in N : 0 < \hat{x}_i < \hat{x}_0\}$ and $L = \{i \in N : \hat{x}_i = 0\}$. To get a contradiction, assume that $M \neq \varnothing$. We construct two distinct feasible solutions $(\tilde{x}_0, \tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}})$ and $(\overline{x}_0, \overline{\boldsymbol{x}}, \overline{\boldsymbol{y}})$ to the Assortment LP such that $(\hat{x}_0, \hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}) = \frac{1}{2}(\tilde{x}_0, \tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}}) + \frac{1}{2}(\overline{x}_0, \overline{\boldsymbol{x}}, \overline{\boldsymbol{y}})$, contradicting the fact that $(\hat{x}_0, \hat{\boldsymbol{x}}, \hat{\boldsymbol{y}})$ is a basic feasible solution. For small $\epsilon > 0$, we define the solution $(\tilde{x}_0, \tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}})$ as

$$\tilde{x}_0 = \hat{x}_0 - V(M)\,\epsilon,$$

$$\tilde{x}_i = \begin{cases} \hat{x}_i - V(M)\,\epsilon & \text{if } i \in H \\ \hat{x}_i + (1 + V(H))\,\epsilon & \text{if } i \in M \\ \hat{x}_i & \text{if } i \in L, \end{cases} \qquad \tilde{y}_{ij} = \begin{cases} \min\{\tilde{x}_i, \tilde{x}_j\} & \text{if } \hat{y}_{ij} = \min\{\hat{x}_i, \hat{x}_j\} \\ \hat{y}_{ij} & \text{if } \hat{y}_{ij} < \min\{\hat{x}_i, \hat{x}_j\}. \end{cases}$$

We claim that $(\tilde{x}_0, \tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}})$ is feasible to the Assortment LP. To see the claim, note that $\tilde{x}_0 + \sum_{i \in N} v_i\,\tilde{x}_i = \hat{x}_0 + \sum_{i \in N} v_i\,\hat{x}_i - V(M)\,\epsilon - \sum_{i \in H} v_i\,V(M)\,\epsilon + \sum_{i \in M} v_i\,(1 + V(H))\,\epsilon = 1$, where the last equality follows by the fact that $\hat{x}_0 + \sum_{i \in N} v_i\,\hat{x}_i = 1$, $\sum_{i \in H} v_i = V(H)$ and $\sum_{i \in M} v_i = V(M)$. Thus, $(\tilde{x}_0, \tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}})$ satisfies the first constraint. Noting that $M \neq \varnothing$, we have $\hat{x}_0 > 0$. By the definitions of $\tilde{x}_i$ and $\tilde{x}_0$, for all $i \in H$, we have $\tilde{x}_i = \hat{x}_i - V(M)\,\epsilon = \hat{x}_0 - V(M)\,\epsilon = \tilde{x}_0$. For all $i \in M$, we have $\hat{x}_i < \hat{x}_0$, so for small $\epsilon > 0$, it follows that $\tilde{x}_i = \hat{x}_i + (1 + V(H))\,\epsilon < \hat{x}_0 - V(M)\,\epsilon = \tilde{x}_0$. Lastly, for all $i \in L$, noting that $\hat{x}_i = 0 < \hat{x}_0$, for small $\epsilon > 0$, we get $\tilde{x}_i = \hat{x}_i < \hat{x}_0 - V(M)\,\epsilon = \tilde{x}_0$. Thus, $(\tilde{x}_0, \tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}})$ satisfies the second constraint as well. If $\hat{y}_{ij} = \min\{\hat{x}_i, \hat{x}_j\}$, then $\tilde{y}_{ij} = \min\{\tilde{x}_i, \tilde{x}_j\}$, so $\tilde{y}_{ij} \leq \tilde{x}_i$ and $\tilde{y}_{ij} \leq \tilde{x}_j$. If, on the other hand, $\hat{y}_{ij} < \min\{\hat{x}_i, \hat{x}_j\}$, then $\hat{y}_{ij} < \min\{\hat{x}_i, \hat{x}_j\} - V(M)\,\epsilon$ for small $\epsilon > 0$. Noting that $\tilde{x}_i \geq \hat{x}_i - V(M)\,\epsilon$ for all $i \in N$, we get $\tilde{y}_{ij} = \hat{y}_{ij} < \min\{\hat{x}_i, \hat{x}_j\} - V(M)\,\epsilon \leq \min\{\tilde{x}_i, \tilde{x}_j\}$, so $\tilde{y}_{ij} \leq \tilde{x}_i$ and $\tilde{y}_{ij} \leq \tilde{x}_j$. Thus, $(\tilde{x}_0, \tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}})$ satisfies the third and fourth constraints. Also, we have $(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}}) \in \mathbb{R}_+^{n+n^2}$ for small $\epsilon > 0$, establishing the claim. We define the solution $(\overline{x}_0, \overline{\boldsymbol{x}}, \overline{\boldsymbol{y}})$ as

$$\overline{x}_0 = \hat{x}_0 + V(M)\,\epsilon,$$

$$\overline{x}_i = \begin{cases} \hat{x}_i + V(M)\,\epsilon & \text{if } i \in H \\ \hat{x}_i - (1 + V(H))\,\epsilon & \text{if } i \in M \\ \hat{x}_i & \text{if } i \in L, \end{cases} \qquad \overline{y}_{ij} = \begin{cases} \min\{\overline{x}_i, \overline{x}_j\} & \text{if } \hat{y}_{ij} = \min\{\hat{x}_i, \hat{x}_j\} \\ \hat{y}_{ij} & \text{if } \hat{y}_{ij} < \min\{\hat{x}_i, \hat{x}_j\}. \end{cases}$$

Using the same argument earlier in this paragraph, we can check that $(\overline{x}_0, \overline{\boldsymbol{x}}, \overline{\boldsymbol{y}})$ is feasible to the Assortment LP. Noting that $M \neq \varnothing$, $V(M) > 0$, so $\tilde{x}_0 \neq \overline{x}_0$, which implies that $(\tilde{x}_0, \tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}})$ and $(\overline{x}_0, \overline{\boldsymbol{x}}, \overline{\boldsymbol{y}})$ are distinct. By the definitions of $(\tilde{x}_0, \tilde{\boldsymbol{x}})$ and $(\overline{x}_0, \overline{\boldsymbol{x}})$, we have $(\hat{x}_0, \hat{\boldsymbol{x}}) = \frac{1}{2}\,(\tilde{x}_0, \tilde{\boldsymbol{x}}) + \frac{1}{2}\,(\overline{x}_0, \overline{\boldsymbol{x}})$, in which case, it only remains to check that $\hat{\boldsymbol{y}} = \frac{1}{2}\,\tilde{\boldsymbol{y}} + \frac{1}{2}\,\overline{\boldsymbol{y}}$.

If we have $\hat{y}_{ij} < \min\{\hat{x}_i, \hat{x}_j\}$, then $\tilde{y}_{ij} = \hat{y}_{ij} = \overline{y}_{ij}$, so $\hat{y}_{ij} = \frac{1}{2}\,\tilde{y}_{ij} + \frac{1}{2}\overline{y}_{ij}$, as desired. Thus, we assume that $\hat{y}_{ij} = \min\{\hat{x}_i, \hat{x}_j\}$. Note that $\tilde{y}_{ij} = \min\{\tilde{x}_i, \tilde{x}_j\}$ in this case. We consider four cases.

<u>Case 1:</u> Assume that $(i, j) \in H \times H$. The definition of H implies that $\hat{x}_i = \hat{x}_j = \hat{x}_0$, so $\hat{y}_{ij} = \min\{\hat{x}_i, \hat{x}_j\} = \hat{x}_0$. Furthermore, if $(i, j) \in H \times H$, then we have $\tilde{x}_i = \hat{x}_i - V(M)\,\epsilon = \hat{x}_0 - V(M)\,\epsilon$ and $\tilde{x}_j = \hat{x}_j - V(M)\,\epsilon = \hat{x}_0 - V(M)\,\epsilon$, so $\tilde{y}_{ij} = \min\{\tilde{x}_i, \tilde{x}_j\} = \hat{x}_0 - V(M)\,\epsilon$. By the symmetric reasoning, we have $\overline{y}_{ij} = \hat{x}_0 + V(M)\,\epsilon$ as well. In this case, we get $\hat{y}_{ij} = \frac{1}{2}\,\tilde{y}_{ij} + \frac{1}{2}\,\overline{y}_{ij}$.

<u>Case 2:</u> Assume that $(i, j) \in (H, M)$. By the definition of H and M, $\hat{x}_i = \hat{x}_0 > \hat{x}_j$, so $\hat{y}_{ij} = \min\{\hat{x}_i, \hat{x}_j\} = \hat{x}_j$. If $(i, j) \in (H, M)$, then we have $\tilde{x}_i = \hat{x}_i - V(M)\,\epsilon$ and $\tilde{x}_j = \hat{x}_j + (1 + V(H))\,\epsilon$, but noting that $\hat{x}_i > \hat{x}_j$, we get $\tilde{x}_i > \tilde{x}_j$ for small $\epsilon > 0$, so $\tilde{y}_{ij} = \min\{\tilde{x}_i, \tilde{x}_j\} = \tilde{x}_j = \hat{x}_j + (1 + V(H))\,\epsilon$. By the symmetric reasoning, we have $\overline{y}_{ij} = \hat{x}_j - (1 + V(H))\,\epsilon$ as well. Thus, $\hat{y}_{ij} = \frac{1}{2}\,\tilde{y}_{ij} + \frac{1}{2}\,\overline{y}_{ij}$.

<u>Case 3:</u> Assume that $(i, j) \in (M, H)$ or $(i, j) \in (M, M)$. In this case, by using the same argument in Case 2, we can show that $\hat{y}_{ij} = \frac{1}{2}\,\tilde{y}_{ij} + \frac{1}{2}\,\overline{y}_{ij}$.

<u>Case 4:</u> Assume that $i \in L$ or $j \in L$. Let $\ell \in \{i, j\}$ be such that $\ell \in L$. The definition of L implies that $\hat{x}_\ell = 0$, so $\hat{y}_{ij} = \min\{\hat{x}_i, \hat{x}_j\} \leq \hat{x}_\ell = 0$. Furthermore, for $\ell \in L$, we have $\tilde{x}_\ell = \hat{x}_\ell = 0$, in which case, we get $\tilde{y}_{ij} = \min\{\tilde{x}_i, \tilde{x}_j\} \leq \tilde{x}_\ell = 0$. By the symmetric reasoning, we have $\overline{y}_{ij} = 0$ as well. In this case, it follows that $\hat{y}_{ij} = 0 = \frac{1}{2}\,\tilde{y}_{ij} + \frac{1}{2}\,\overline{y}_{ij}$. $\square$

C.3 Proof of Lemma 4.5

In this section, we give a proof for Lemma 4.5. By the discussion at the beginning of Section 4.6, recall that if the Compact LP has multiple optimal solutions, then we choose the one that has the largest value for the decision variable x_0. Furthermore, to obtain a solution that has the largest value for the decision variable for x_0, for $\epsilon > 0$, we can add the additional term $\epsilon\, x_0$ to the objective function of the Compact LP. If ϵ is small enough, then solving the Compact LP with the additional term provides an optimal solution to the original version of the Compact LP that has the largest value for the decision variable x_0. Thus, we consider the problem

$$\max_{(x_0,\boldsymbol{x},\boldsymbol{y})\in\mathbb{R}\times\mathbb{R}_+^{n+n^2}} \Bigg\{ T\sum_{i\in N} r_i \Bigg((v_i + \lambda\,\theta_i)\, x_i + \lambda\,\theta_i \sum_{j\in N} v_j\, y_{ij} \Bigg) + \epsilon\, x_0 \ : \tag{C.2}$$
$$T\sum_{i\in N} a_{qi} \Bigg((v_i + \lambda\,\theta_i)\, x_i + \lambda\,\theta_i \sum_{j\in N} v_j\, y_{ij} \Bigg) \leq c_q \ \ \forall\, q \in M,$$
$$x_0 + \sum_{i\in N} v_i\, x_i = 1,$$
$$x_i \leq x_0 \ \ \forall\, i \in N,$$
$$y_{ij} \leq x_i \ \ \forall\, i,j \in N, \ \ y_{ij} \leq x_j \ \ \forall\, i,j \in N \Bigg\}.$$

If ϵ is small enough, then a basic optimal solution to the problem above is also a basic optimal solution to problem the Compact LP. So, it is enough to show that if $(x_0^*, \boldsymbol{x}^*, \boldsymbol{y}^*)$ is a basic optimal solution to problem (C.2), then we have $y_{ij}^* = \min\{x_i^*, x_j^*\}$ for all $i, j \in N$. For notational brevity, we let $\mathcal{P} = \{(x_0, \boldsymbol{x}, \boldsymbol{y}) \in \mathbb{R}\times\mathbb{R}_+^{n+n^2} : x_0 + \sum_{i\in N} v_i\, x_i = 1, \ x_i \leq x_0 \ \forall\, i \in N, \ y_{ij} \leq \min\{x_i, x_j\} \ \forall\, i,j \in N\}$, denoting the polytope captured by the last four constraints in the LP above. The proof of Lemma 4.5 uses two lemmas, which closely resemble results already established in the main text.

In the first lemma, we consider a slightly modified version the Assortment LP, where we add the additional term $\epsilon\, x_0$ to the objective function. In particular, consider the LP

$$\max_{(x_0,\boldsymbol{x},\boldsymbol{y})\in\mathbb{R}\times\mathbb{R}_+^{n+n^2}} \left\{ \sum_{i\in N} r_i \left((v_i + \lambda\,\theta_i)\, x_i + \lambda\,\theta_i \sum_{j\in N} v_j\, y_{ij} \right) + \epsilon\, x_0 \;\;:\;\; (x_0, \boldsymbol{x}, \boldsymbol{y}) \in \mathcal{P} \right\}. \tag{C.3}$$

In the next lemma, we relate an optimal solution to the LP above to an optimal solution of a slightly modified version of the Mixture problem.

Lemma C.1. *For a basic optimal solution $(x_0^*, \boldsymbol{x}^*, \boldsymbol{y}^*)$ to problem $(C.3)$, let $S^* = \{i \in N : x_i^* > 0\}$. Then, S^* is an optimal solution to the problem*

$$\max_{S\subseteq N} \left\{ \sum_{i\in S} r_i \left(\frac{v_i}{1+V(S)} + \lambda\,\theta_i \right) + \frac{\epsilon}{1+V(S)} \right\}. \tag{C.4}$$

Lemma C.1 is an analogue of Theorem 4.2 and its proof follows the same line of reasoning that we used in the proof of Theorem 4.2. We skip the proof.

In the second lemma, we relate problem (C.2) to a slightly modified version of the Choice-Based LP. We can view this lemma as an analogue of Theorem 4.4.

Lemma C.2. *Consider the* Choice-Based LP *after adding the term $\sum_{S\subseteq N} \frac{\epsilon}{1+V(S)} w(S)$ to the objective function, which is given by*

$$\begin{aligned} \max_{\boldsymbol{w}\in\mathbb{R}_+^{2^n}} \Bigg\{ T \sum_{S\subseteq N} \sum_{i\in S} r_i \left(\frac{v_i}{1+V(S)} + \lambda\,\theta_i \right) w(S) + \sum_{S\subseteq N} \frac{\epsilon}{1+V(S)}\, w(S) \;\; : \\ T \sum_{S\subseteq N} \sum_{i\in S} a_{qi} \left(\frac{v_i}{1+V(S)} + \lambda\,\theta_i \right) w(S) \le c_q \quad \forall\, q \in M, \\ \sum_{S\subseteq N} w(S) = 1 \Bigg\}. \end{aligned} \tag{C.5}$$

Then, the optimal objective values of problems $(C.2)$ and $(C.5)$ are the same. Furthermore, the optimal values of the dual variables for the first constraint in problems $(C.2)$ and $(C.5)$ are the same.

The proof of the lemma above uses the same reasoning that we use in the proof of Theorem 4.4 in conjunction with Lemma C.1. We skip the proof.

Next, using the dual variables $\boldsymbol{\mu} = \{\mu_q : q \in M\}$, π, $\boldsymbol{\alpha} = \{\alpha_i : i \in N\}$, $\boldsymbol{\eta} = \{\eta_{ij} : i, j \in N\}$ and $\boldsymbol{\sigma} = \{\sigma_{ij} : i, j \in N\}$, the dual of problem (C.2) is given by

$$\min_{\substack{(\boldsymbol{\mu},\pi,\boldsymbol{\alpha},\boldsymbol{\eta},\boldsymbol{\sigma})\in \\ \mathbb{R}_+^m\times\mathbb{R}\times\mathbb{R}_+^{n+2n^2}}} \Bigg\{ \sum_{q\in M} c_q\,\mu_q + \pi \;\; : \tag{C.6}$$

$$\pi = \sum_{i\in N} \alpha_i + \epsilon,$$

$$v_i\,\pi + \alpha_i - \sum_{j\in N} \eta_{ij} - \sum_{j\in N} \sigma_{ji} \geq T\,(v_i + \lambda\,\theta_i) \left(r_i - \sum_{q\in M} a_{qi}\,\mu_q \right) \quad \forall\, i \in N,$$

$$\eta_{ij} + \sigma_{ij} \geq T\,\lambda\,\theta_i\,v_j \left(r_i - \sum_{q\in M} a_{qi}\,\mu_q \right) \quad \forall\, i, j \in N \Bigg\}.$$

In the next lemma, we use complementary slackness to give two useful properties that are satisfied by an optimal primal-dual solution pair for problem (C.2).

Lemma C.3. *Let $(x_0^*, \boldsymbol{x}^*, \boldsymbol{y}^*)$ and $(\boldsymbol{\mu}^*, \pi^*, \boldsymbol{\alpha}^*, \boldsymbol{\eta}^*, \boldsymbol{\sigma}^*)$ be a basic optimal primal-dual solution pair for problem $(C.2)$ and $S^* = \{i \in N : x_i^* > 0\}$. Then, we have*

$$\pi^* = \sum_{i\in S^*} \alpha_i^* + \epsilon,$$

$$\sum_{i\in S^*}\sum_{j\in N} (\eta_{ij}^* + \sigma_{ji}^*) = \sum_{i\in S^*}\sum_{j\in S^*} (\eta_{ij}^* + \sigma_{ij}^*).$$

Proof. To see the first equality, note that $x_0^* > 0$. Otherwise, we have $x_i^* = 0$ for all $i \in N$ by the third constraint in problem (C.2), in which case, it is impossible to satisfy the second constraint. Since $x_0^* > 0$ and $x_i^* = 0$ for all $i \notin S^*$, using complementary slackness on the third constraint in problem (C.2), we have $\alpha_i^* = 0$ for all $i \notin S^*$, in which case, by the first constraint in problem (C.6), we get $\pi^* = \sum_{i \in S^*} \alpha_i^* + \epsilon$. To see the second equality, if $i \in S^*$ and $j \notin S^*$, then $x_i^* > 0$ and $x_j^* = 0$, in which case, by the last two constraints in problem (C.2), we have $y_{ij}^* = 0$ and $y_{ji}^* = 0$. Therefore, we get $y_{ij}^* < x_i^*$ and $y_{ji}^* < x_i^*$, so using complementary slackness on the last two constraints in problem (C.2), we get $\eta_{ij}^* = 0$ and $\sigma_{ji}^* = 0$. Thus, if $i \in S^*$ and $j \notin S^*$, then $\eta_{ij}^* = 0$ and $\sigma_{ji}^* = 0$. In this case, the second equality in the lemma follows by noting that

$$\begin{aligned}\sum_{i \in S^*} \sum_{j \in N} (\eta_{ij}^* + \sigma_{ji}^*) \;&=\; \sum_{i \in S^*} \sum_{j \in S^*} (\eta_{ij}^* + \sigma_{ji}^*) \;+\; \sum_{i \in S^*} \sum_{j \notin S^*} (\eta_{ij}^* + \sigma_{ji}^*) \\ &=\; \sum_{i \in S^*} \sum_{j \in S^*} (\eta_{ij}^* + \sigma_{ji}^*) \;=\; \sum_{i \in S^*} \sum_{j \in S^*} (\eta_{ij}^* + \sigma_{ij}^*).\end{aligned}$$

□

Proof of Lemma 4.5. Let $(x_0^*, \boldsymbol{x}^*, \boldsymbol{y}^*)$ and $(\boldsymbol{\mu}^*, \pi^*, \boldsymbol{\alpha}^*, \boldsymbol{\eta}^*, \boldsymbol{\sigma}^*)$ be a basic optimal primal-dual solution pair for problem (C.2). By the discussion right after problem (C.2), it is enough show that $y_{ij}^* = \min\{x_i^*, x_j^*\}$ for all $i, j \in N$. Let $S^* = \{i \in N : x_i^* > 0\}$. Consider $i, j \in S^*$. We have $x_i^* > 0$ and $x_j^* > 0$, so using complementary slackness on the last two constraints in problem (C.2), if $y_{ij}^* = 0$, then $\eta_{ij}^* = 0$ and $\sigma_{ij}^* = 0$. On the other hand, if $y_{ij}^* > 0$, then using complementary slackness on the last constraint in problem (C.6), we have $\eta_{ij}^* + \sigma_{ij}^* = T\,\lambda\,\theta_i\,v_j\,(r_i - \sum_{q \in M} a_{qi}\,\mu_q^*)$. Therefore, for all $i, j \in S^*$, we have $\eta_{ij}^* + \sigma_{ij}^* \leq T\,\lambda\,\theta_i\,v_j\,\left(r_i - \sum_{q \in M} a_{qi}\,\mu_q^*\right)^+$, where we let $(a)^+ = \max\{a, 0\}$.

For all $i \in S^*$, $x_i^* > 0$, so using complementary slackness on the second constraint in problem (C.6), this constraint holds as equality for all $i \in S^*$. Adding over all $i \in S^*$ yields

$$\begin{aligned}&\sum_{i \in S^*} v_i\,\pi^* + \sum_{i \in S^*} \alpha_i^* \\ &=\; T \sum_{i \in S^*} (v_i + \lambda\,\theta_i) \left(r_i - \sum_{q \in M} a_{qi}\,\mu_q^* \right) + \sum_{i \in S^*} \sum_{j \in N} \eta_{ij}^* + \sum_{i \in S^*} \sum_{j \in N} \sigma_{ji}^* \\ &\stackrel{(a)}{=}\; T \sum_{i \in S^*} (v_i + \lambda\,\theta_i) \left(r_i - \sum_{q \in M} a_{qi}\,\mu_q^* \right) + \sum_{i \in S^*} \sum_{j \in S^*} (\eta_{ij}^* + \sigma_{ij}^*)\end{aligned}$$

$$\stackrel{(b)}{\leq} T \sum_{i\in S^*} (v_i + \lambda\,\theta_i) \left(r_i - \sum_{q\in M} a_{qi}\,\mu_q^* \right) + T \sum_{i\in S^*}\sum_{j\in S^*} \lambda\,\theta_i\, v_j \left(r_i - \sum_{q\in M} a_{qi}\,\mu_q^* \right)^+$$
$$\stackrel{(c)}{\leq} T \sum_{i\in S^*} v_i \left(r_i - \sum_{q\in M} a_{qi}\,\mu_q^* \right)^+ + T \sum_{i\in S^*} \lambda\,\theta_i\,(1+V(S^*)) \left(r_i - \sum_{q\in M} a_{qi}\,\mu_q^* \right)^+,$$

where (a) follows from Lemma C.3, (b) holds since $\eta_{ij}^*+\sigma_{ij}^* \leq T\,\lambda\,\theta_i\,v_j\,\big(r_i - \sum_{q\in M} a_{qi}\,\mu_q^*\big)^+$ as in the previous paragraph and (c) holds by arranging the terms and noting that $\sum_{j\in S^*} v_j = V(S^*)$.

The expression on the left side of the chain of inequalities above is given by $\sum_{i\in S^*} v_i\,\pi + \sum_{i\in S^*} \alpha_i^* = V(S^*)\,\pi^* + \sum_{i\in S^*} \alpha_i^* = (1+V(S^*))\,\pi^* - \epsilon$, where the last equality follows from Lemma C.3. In this case, replacing the left side of the chain of inequalities above by $(1+V(S^*))\,\pi^* - \epsilon$ and dividing both sides of the inequality by $1+V(S^*)$, we get

$$\pi^* - \frac{\epsilon}{1+V(S^*)} \leq T \sum_{i\in S^*} \left(r_i - \sum_{q\in M} a_{qi}\,\mu_q^* \right)^+ \left(\frac{v_i}{1+V(S^*)} + \lambda\,\theta_i \right). \tag{C.7}$$

To show the result by contradiction, assume that there exists $i,j \in N$ such that $y_{ij}^* < \min\{x_i^*, x_j^*\}$. Since $y_{ij}^* \geq 0$, it must be the case that $x_i^* > 0$ and $x_j^* > 0$, so we get $i,j \in S^*$.

Letting $i,j \in S^*$ such that $y_{ij}^* < \min\{x_i^*, x_j^*\}$, using complementary slackness on the last two constraints in problem (C.2), it follows that $\eta_{ij}^* = 0$ and $\sigma_{ij}^* = 0$, in which case, by the last constraint in problem (C.6), we have $0 \geq r_i - \sum_{q\in M} a_{qi}\,\mu_q^*$. Thus, there exists $i \in S^*$ such that $r_i - \sum_{q\in M} a_{qi}\,\mu_q^* \leq 0$. Let $N^* = \{i \in S^* : r_i - \sum_{q\in M} a_{qi}\,\mu_q^* \leq 0\}$, so N^* is non-empty.

By Lemma C.2, problems (C.2) and (C.5) have the same optimal objective values. Letting z_{LP}^* be their common optimal objective value, since problem (C.6) is the dual of (C.2), we get

$$z_{\text{LP}}^* = \sum_{q\in M} c_q\,\mu_q^* + \pi^*$$
$$\stackrel{(c)}{\leq} \sum_{q\in M} c_q\,\mu_q^* + T \sum_{i\in S^*} \left(r_i - \sum_{q\in M} a_{qi}\,\mu_q^* \right)^+ \left(\frac{v_i}{1+V(S^*)} + \lambda\,\theta_i \right) + \frac{\epsilon}{1+V(S^*)}$$

$$
\begin{aligned}
&\overset{(d)}{=} \sum_{q \in M} c_q\, \mu_q^* + T \sum_{i \in S^* \setminus N^*} \left(r_i - \sum_{q \in M} a_{qi}\, \mu_q^* \right) \left(\frac{v_i}{1 + V(S^*)} + \lambda\, \theta_i \right) + \frac{\epsilon}{1 + V(S^*)} \\
&\overset{(e)}{<} \sum_{q \in M} c_q\, \mu_q^* + T \sum_{i \in S^* \setminus N^*} \left(r_i - \sum_{q \in M} a_{qi}\, \mu_q^* \right) \left(\frac{v_i}{1 + V(S^* \setminus N^*)} + \lambda\, \theta_i \right) + \frac{\epsilon}{1 + V(S^* \setminus N^*)} \\
&\leq \sum_{q \in M} c_q\, \mu_q^* + \max_{S \subseteq N} \left\{ T \sum_{i \in S} \left(r_i - \sum_{q \in M} a_{qi}\, \mu_q^* \right) \left(\frac{v_i}{1 + V(S)} + \lambda\, \theta_i \right) + \frac{\epsilon}{1 + V(S)} \right\} \\
&\overset{(f)}{=} \sum_{q \in M} c_q\, \mu_q^* + \max_{\boldsymbol{w} \in \mathbb{R}_+^{2n}} \Bigg\{ T \sum_{S \subseteq N} \sum_{i \in S} \left(r_i - \sum_{q \in M} a_{qi}\, \mu_q^* \right) \left(\frac{v_i}{1 + V(S)} + \lambda\, \theta_i \right) w(s) \\
&\qquad\qquad\qquad\qquad + \sum_{S \subseteq N} \frac{\epsilon}{1 + V(S)}\, w(s) \;:\; \sum_{S \subseteq N} w(S) = 1 \Bigg\},
\end{aligned}
$$

where (c) uses (C.7), (d) holds since $r_i - \sum_{q \in M} a_{qi}\, \mu_q^* \leq 0$ for all $i \in N^*$, (e) holds since $N^* \neq \varnothing$, so $V(S^* \setminus N^*) < V(S^*)$ and (f) holds by randomizing instead of picking one assortment.

Consider computing the dual function for problem (C.5) by associating the dual multipliers $\boldsymbol{\mu}^* = \{\mu_q^* : q \in M\}$ with the first constraint in this problem. In this case, the value of the dual function precisely corresponds to the expression on the right side of the chain of inequalities above. Furthermore, by Lemma C.2, $\boldsymbol{\mu}^*$, which gives the optimal values of the dual variables associated with the first constraint in problem (C.2), also gives the optimal values of the dual variables associated with the first constraint in problem (C.5). Thus, the expression on the right side of the chain of inequalities above is the optimal objective value of problem (C.5), which is also z^*_{LP}. In this case, noting the strict inequality in (e), we get a contradiction. $\square$

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